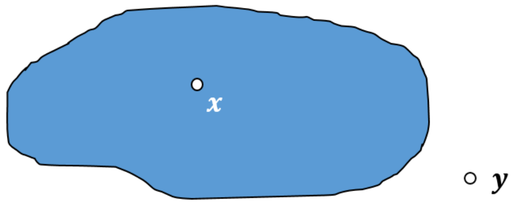
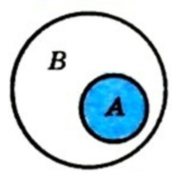
# To'plamlar

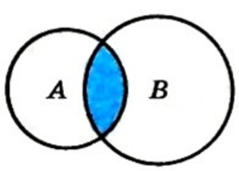
Birorta ham elementga ega bo‘lmagan to‘plam **bo‘sh to‘plam** deyiladi va Ø kabi belgilanadi.

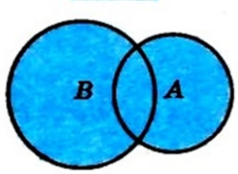
$x \in A -$ *x element* $A$ to‘plamning elementi yoki $A$ to‘plamga tegishli ekanligini bildiradi. Masalan. $A$ $-$juft sonlar to'plami bo'lsa $2 \in A$.

$y \notin A -$ yozuv - *y element* $A$ to‘plamga tegishli emasligini bildiradi. Masalan. $A$ $-$juft sonlar to'plami bo'lsa $1 \notin A$.

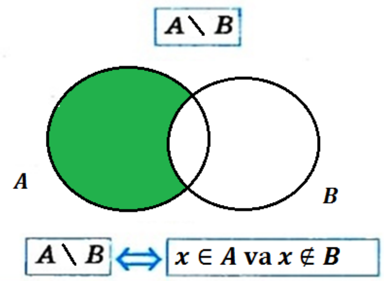



| QISM TO'PLAM | |
|---|---|
| $\boxed{A \subset B}$   $\boxed{A \subset B} \Longleftrightarrow \boxed{\forall x \in A \Rightarrow x \in B}$ | Agar $A$ to‘plamning har bir elementlari $B$ to‘plamga tegishli bo‘lsa, $A$ to‘plam $B$ to‘plamning **qism to‘plami** deyiladi va $A \subseteq B$ kabi yoziladi. Bunday holatda “ $A$ to‘plam $B$ da yotadi” yoki “ $A$ to‘plam $B$ ning qismi” deb ham yuritiladi.<br>Masalan: $A$ $-$juft sonlar to'plami:<br>$A \subset \mathbb{N}$. |

| TO'PLAMLARNING KESISHMASI | |
|---|---|
| $\boxed{A \cap B}$   $\boxed{A \cap B} \Longleftrightarrow \boxed{x \in A \text{ va } x \in B}$ | ***A* va *B* to‘plamlarning kesishmasi** deb, shunday to‘plamga aytiladiki, bu to‘plamning elementlari ham A to‘plamga ham B to‘plamga tegishli bo'ladi.<br>Masalan: $A = \{2,4,6,8, \dots\}$ $-$juft sonlar to'plami<br>$B = \{2,3,5,7,11, \dots\}$ $-$tub sonlar to'plami:<br>$A \cap B = \{2\}$. |

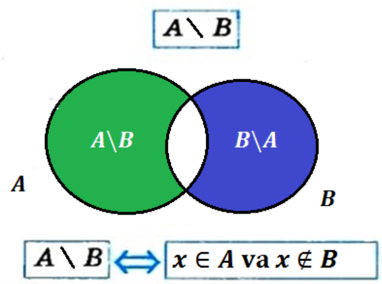
| TO'PLAMLARNING BIRLASHMASI | |
|---|---|
| $\boxed{A \cup B}$   $\boxed{A \cup B} \Longleftrightarrow \boxed{x \in A \text{ yoki } x \in B}$ | ***A* va *B* to‘plamlarning birlashmasi** deb, shunday to‘plamga aytiladiki, bu to‘plamning elementlari yoki A to‘plamda, yoki B to‘plamda yotadi.<br>Masalan: $A = \{2,4,6,8, \dots\}$ $-$juft sonlar to'plami<br>$B = \{1,2,5,7,9, \dots\}$ $-$toq sonlar to'plami:<br>$A \cup B = \mathbb{N}$. |

## TO’PLAMLARNING AYIRMASI



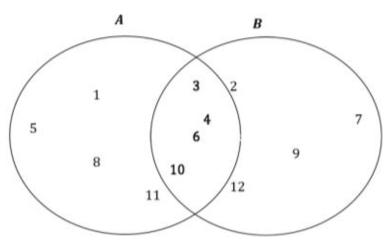

***A* va *B* to‘plamlarning ayirmasi** deb, ***A*** ning ***B*** ga tegishli bo‘lmagan barcha elementlaridan tuzilgan to‘plamga aytiladi.
Masalan: $A = \{2,4,6,8, \ldots\}$ –juft sonlar to’plami
$B = \{2,3,5,7,11, \ldots\}$ –tub sonlar to’plami:
$$A \backslash B = \{4,6,8, \ldots\}$$

## TO’PLAMLARNING SIMMIRTIKAYIRMASI



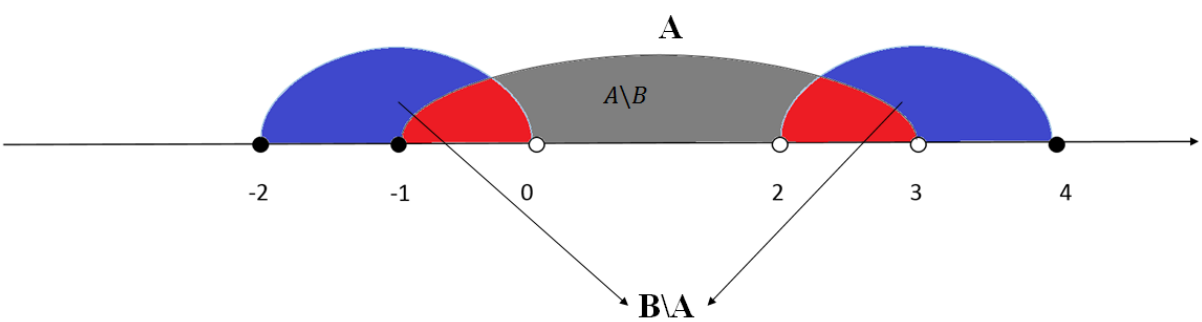

***A* va *B* to‘plamlarning ayirmasi** deb, ***A*** ning ***B*** ga tegishli bo‘lmagan barcha elementlaridan tuzilgan to‘plamga aytiladi.
Masalan: $A = \{2,4,6,8, \ldots\}$ –juft sonlar to’plami
$B = \{2,3,5,7,11, \ldots\}$ –tub sonlar to’plami:
$$A \backslash B = \{4,6,8, \ldots\}$$

**1-misol**. Berilgan $A$ va $B$ to’plamlar uchun $\mathrm{A} \cap \mathrm{B}, \mathrm{A} \cup \mathrm{B}, \mathrm{A} \backslash \mathrm{B}$ va $\mathrm{B} \backslash \mathrm{A}$ to’plamlarni toping: $A = \{1,3,4,5,6,8,10,11\}$, $B = \{2,3,4,6,7,9,10,12\}$.



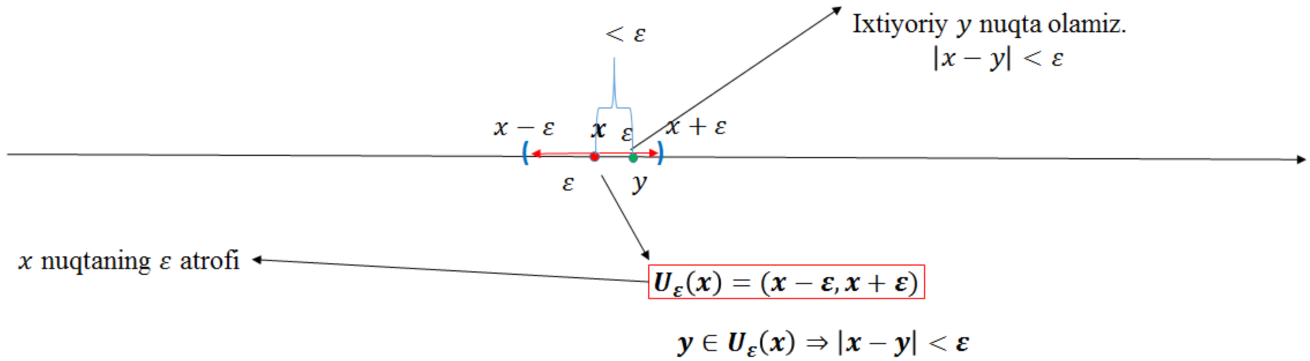

$\mathrm{A} \cap \mathrm{B} = \{3,4,6,10\}$, $\mathrm{A} \cup \mathrm{B} = \{1,2,3,4,5,6,7,8,9,10,11,12\}$, $\mathrm{A} \backslash \mathrm{B} = \{1,5,8,11\}$,
$\mathrm{B} \backslash \mathrm{A} = \{2,7,9,12\}$.

**2-misol**. Berilgan $A$ va $B$ to’plamlar uchun $\mathrm{A} \cap \mathrm{B}$, $\mathrm{A} \cup \mathrm{B}$, $\mathrm{A} \backslash \mathrm{B}$ va $\mathrm{B} \backslash \mathrm{A}$ to’plamlarni toping: $A = [-1; 3), B = [-2; 0) \cup (2; 4]$.

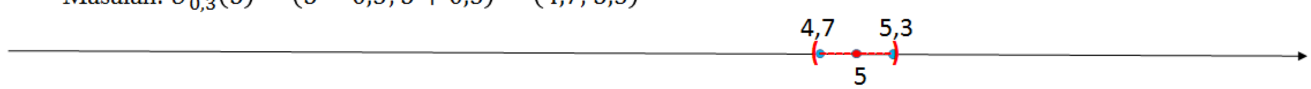



$A \cap B = [-1; 0) \cup (2; 3)$, $A \cup B = [-2; 4]$, $A \backslash B = [0; 2]$,
$B \backslash A = [-2; -1) \cup [3; 4]$ .

$x_0 \in \mathbb{R}$ nuqtaning $\varepsilon > 0$ artofi deb $(x_0 - \varepsilon, x_0 + \varepsilon)$ to'plamga aytiladi.




Masalan: $U_{0,3}(5) = (5-0{,}3; 5+0{,}3) = (4{,}7; 5{,}3)$




Quyidagi belgilashlar ham ko'p qo'llaniladi:

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \ldots \cup A_n, \qquad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \ldots \cap A_n.$$

**Masalalar.**

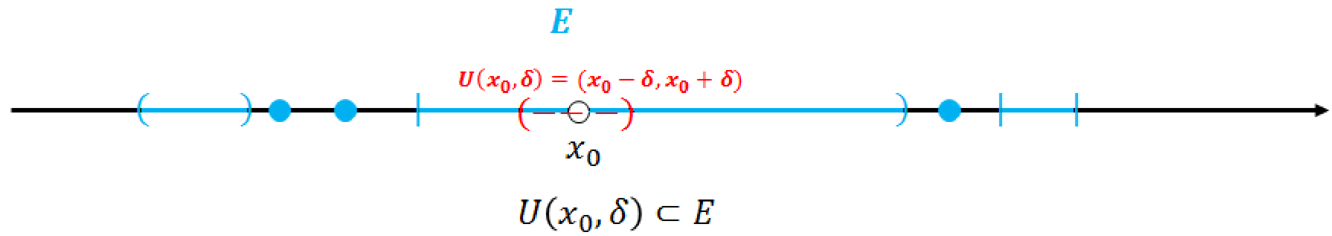
1. $A = \{3n+1 : n \in \mathbb{N}\}, B = \{2n+1 : n \in \mathbb{N}\}$ bo'lsa $A \cap B$ toping.

Berilgan $A$ va $B$ to'plamlar uchun $\mathrm{A} \cap \mathrm{B}$, $\mathrm{A} \cup \mathrm{B}$, $\mathrm{A} \backslash \mathrm{B}$ va $\mathrm{B} \backslash \mathrm{A}$ to'plamlarni toping va uni tasvirlang.

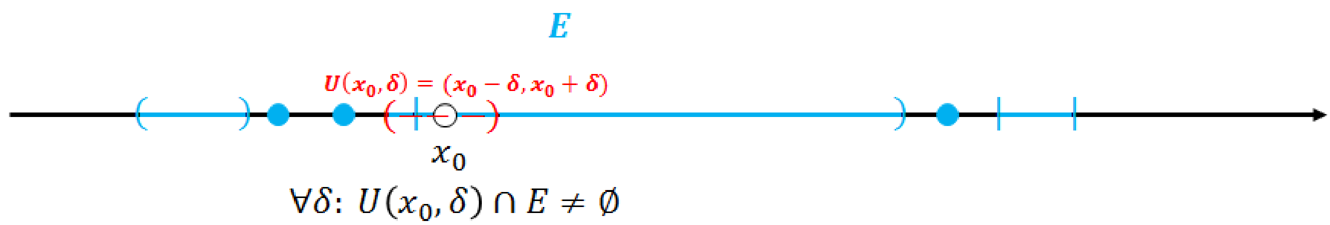
2. $A = [-3; 3), B = [-1; 0) \cup (2; 4]$;
3. $A = [-2; 5) \cup [4; 7),\ B = [-3; 1) \cup (2; 4]$;
4. $\bigcup_{n=1}^{100} [-n, n+1), \bigcap_{n=1}^{100} [-n, n]$;
5. $A = \{(x,y) : x^2 + y^2 \le 1\}, B = \{(x,y) : |x| + |y| \le 1\}$ ;
6. $A = \{(x,y) : \max\{|x|, |y|\} \le 1\}, B = \{(x,y) : |x| + |y| \le 1\}$;
7. $A = \left\{x : \sin\frac{x}{3} = 1\right\}, B = \left\{x : \sin\frac{x}{2} = 1\right\}$.
8. $\{(x,y) : |x| + |y| + |x+y| \le 2\} \subset \{(x,y) : \max\{|x|, |y|\} \le 1\}$ bo'lishini isbotlang va ularni tekislikda tasvirlang.

## Ochiq va yopiq to'plamlar

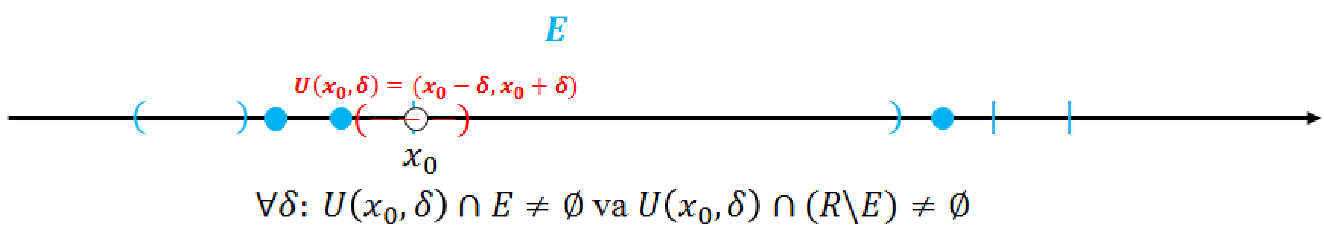
$\mathbb{R}$ to'plamda $\forall x, y \in \mathbb{R}$ nuqtalar orasidagi masofa deb $p(x,y) = |x-y|$ qiymatga aytiladi. $U(a,\varepsilon) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$ to'plamga $a$ nuqtani $\varepsilon$ atrofi deyiladi. Agar $E \subset \mathbb{R}$ to'plam $x_0 \in E$ nuqta uchun shunday $U(x_0, \delta) \subset E$ atrof mavjud bo'lsa $x_0$ nuqta $E$ to'plamning ichki nuqta deyiladi.




Agar $x_0$ nuqtada $\forall\delta>0$ son uchun $U\left(x_0,\delta\right)\cap E\neq\varnothing$ bo'lsa $x_0-E$ to'plamning urunish nuqtasi deyiladi.




Agar $x_0$ nuqta va $\forall\delta>0$ son uchun $U\left(x^0,\delta\right)\cap E\neq\varnothing$ va $U\left(x_0,\delta\right)\cap\left(\mathbb{R}\setminus E\right)\neq\varnothing$ bo'lsa u holda $x_0-E$ to'plamning chegaraviy nuqtasi deyiladi.




Barcha nuqtalari ichki nuqta bo'lgan $E\subset\mathbb{R}$ to'plamga ochiq to'plam deyildi. Agar $E\subset\mathbb{R}$ to'plam uchun $\mathbb{R}\setminus E-$ochiq bo'lsa $E-$yopiq to'plam deyiladi. $E$ to'plamning barcha ichki nuqtalari to'plami $\overset{\circ}{E}$, urunish nuqtalari to'plami $\bar{E}$ va chegaraviy nuqtalari to'plami $\partial E$ orqali belgilaymiz. Agar $E$ to'plam uchun $\exists U\left(0,R\right)$ shar topilib $E\subset U\left(0,R\right)$ bo'lsa u holda $E-$to'plam chegaralangan aks holda chegaralanmagan deyiladi.

**Masalalar**

1. $\left(a,b\right)$ to'plam ochiq, $\left[a,b\right]$ to'plam esa yopiqligini isbotlang.
2. $\left|e^x-2\right|\leq 1$ tengsizlikni yechimlar to'plami yopiq to'plam ekanligini isbotlang.
3. $x^3-3x+2<0$ tengsizlikni yechimlar to'plami ochiq to'plam ekanligini isbotlang.
4. $A=\left\{\dfrac{1}{n}:n\in\mathbb{N}\right\}$ to'plam uchun $\overset{\circ}{A}$, $\bar{A}$ va $\partial A$ to'plamlarni toping.

5. $A=\left\{x\in R:\left\{\frac{x}{4}\right\}<\frac{1}{2}\right\}$ to'plam uchun $\overset{\circ}{A}$, $\bar{A}$ va $\partial A$ to'plamlarni toping.

6. $A, B-$ochiq to'plamlar bo'lsa $A\cup B$ va $A\cap B$ to'plamlar ham ochiq to'plam bo'lishini isbotlang.

7. Ixtiyoriy $A\subset\mathbb{R}$ to'plam uchun $\partial A$ va $\bar{A}$ to'plamlar yopiq to'plam bo'lishini isbotlang.

8. $x^x>3$ tengsizlikni yechimlari to'plami ochiq to'plam ekanligini isbotlang.

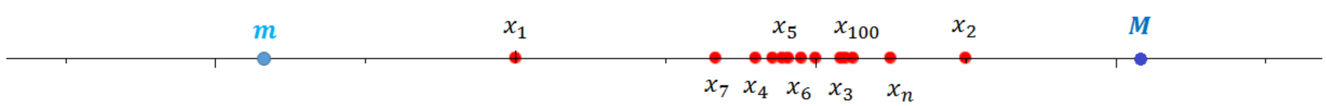
## Ketma-ketliklar

$f:\mathbb{N}\to\mathbb{R}$ akslantirishga sonli ketma-ketlik deyiladi. Odatda sonli ketma-ketliklar $a_1,a_2,a_3,...,a_n,...$ yoki $\{a_n\}_{n=1}^{\infty}$ yoki $\{a_n\}$ orqali belgilanadi, bu yerda $a_n=f(n)$.

Agar shunday $\exists n, M$ haqiqiy sonlar mavjud bo'lib barcha $n$ natural son uchun $m\le a_n\le M$ tengsizlik bajarilsa $\{a_n\}$ sonli ketma-ketlik chegaralangan deyiladi. Buni qisqacha quyidagicha yozish ham mumkin:

$$\exists N>0:|a_n|\le N, n\in\mathbb{N},$$

bu yerda $N=\max\{|m|,|M|\}$.




Agar $\forall N\in\mathbb{N}$ natural son uchun $\{a_n\}$ sonli ketma-ketlikni shunday $\exists a_{n'}$ xadi topilib $|a_{n(N)}|\ge N$ tengsizlik bajarilsa u holda $\{a_n\}$ ketma-ketlik chegaralangan deyiladi.

Ketma-ketlik chegaralanganligini isbotlang.

Na'muna:

**1-misol**. $x_n=\log_{n+1}2$ .

Ma'lumki $x_n=\log_{n+1}2>\log_{n+1}1=0$ va $x_n=\frac{\ln 2}{\ln(n+1)}\le\frac{\ln 2}{\ln(1+1)}=1$. Demak $0<x_n\le 1$ bo'lib $\{x_n\}-$ketma-ketlik chegaralangan.

**2-misol**. $x_n=1+\frac{1}{2^a}+\frac{1}{3^a}+...+\frac{1}{n^a},(a\ge 2)$.

$n \geq 2$ uchun $x_n = \frac{1}{n^a} \leq \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ tengsizlik o'rinli. Bu tengsizlikdan foydalanib

$$x_n = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \ldots + \frac{1}{n^a} < 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n-1} - \frac{1}{n} = 2 - \frac{1}{n} < 2$$

tengsizlikka ega bo'lamiz. Boshqa tomondan $x_n \geq 1$ ekanligini ko'rsatish oson. Demak $1 < x_n < 2$ bo'lib $x_n -$ketma-ketlik chegaralangan.

**3-misol**. $x_n = \begin{cases} \sqrt{n}, n - \text{ juft} \\ \frac{1}{n}, n - toq \end{cases}$.

Faraz qilaylik $x_n$ chegaralangan. U holda shunday $M$ natural son topiladiki $|x_n| \leq M, \forall n \in \mathbb{N}$ o'rinli. Xususan $n = 4M^2$ uchun ham. Ammo bu holda

$x_{4M^2} = \sqrt{4M^2} = 2M > M$ bo'lib chegaralangaligiga zid. Demak $x_n -$ketma-ketlik chegaralanmagan.

**4-misol**. $x_n = \log_2\left(n^2 + n\right)$ .

Agar $n = 2^M, M \in \mathbb{N}$ bo'lsa u holda

$$x_n = \log_2\left(n^2 + n\right) > \log_2 n^2 = \log_2 2^{2M} = 2M > M$$

bo'lib $x_n -$ketma-ketlik har qanday natural sondan katta hadi bor. Demak $x_n -$ketma-ketlik chegaralanmagan.

Masalalar.

Sonli ketma-ketlikni chegaralanganligini isbotlang.

1. $a_n = 3\sin n;$ 2. $a_n = 3\sin n + 4\cos n;$ 3. $a_n = \frac{1}{1+n^2};$ 4. $a_n = \frac{n}{n+1};$

5. $a_n = \sqrt{n+1} - \sqrt{n}$ ; 6. $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}$ .

Sonli ketma-ketlikni chegaralanmaganligini isbotlang.

1. $a_n = \sqrt[3]{n};$ 2. $a_n = \ln n;$ 3. $a_n = n^{(-1)^n};$

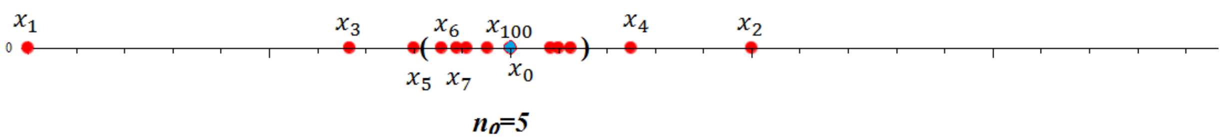
4. $a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}}.$

## Limitlar

$\{x_n\}$ – sonli ketma-ketlik berilgan bo'lsin.

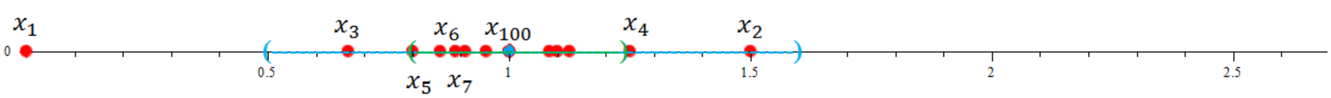
**Ta'rif**. Agar ixtoyoriy $\forall \varepsilon > 0$ musbat son olganda ham shunday $n_0(\varepsilon) \in \mathbb{N}$ son topilib $n_0$ sondan boshlab barcha $n > n_0$ natural sonlar uchun $|x_n - x_0| < \varepsilon$ tengsizlik bajarilsa (ya'ni $n_0$ sondan boshlab $\{x_n\}$ ketma-ketlikning barcha barcha hadi $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$ oraliqda yotsa) u holda $x_0$ soni $\{x_n\}$ ketma-ketlikni limiti deyiladi va quyidagicha belgilanadi:

$$\lim_{n\to\infty} x_n = x_0.$$




**Misol**. $x_n = \frac{(-1)^n + n}{n}$ ketma-ketlikni limiti $x_0 = 1$ ekanligini isbotlang.

Buni isbotlash uchun $x_0 = 1$ nuqtaning turli atroflarini ko'ramiz va bu atrofga qaysi raqamdan boshlab tushishini ko'ramiz.




$U_{0,5}(1) = (0{,}5; 1{,}5)$ atrofini ko'ramiz. $x_3, x_4, x_5, \dots, x_{100}, \dots \in U_{0,5}(1)$. $n \geq 3$ uchun $x_n \in U_{0,5}(1)$, ya'ni $|x_n - 1| < 0{,}5$. $U_{0,2}(1) = (0{,}8; 1{,}2)$ atrofini ko'ramiz. $x_6, x_7, x_8, \dots, x_{100}, \dots \in U_{0,2}(1)$. $n \geq 6$ uchun $x_n \in U_{0,2}(1)$ ya'ni $|x_n - 1| < 0{,}2$, $U_{0,05}(1) = (0{,}95; 1{,}05)$ atrofini ko'ramiz. $a_{21}, a_{22}, a_{23}, \dots, a_{100}, \dots \in U_{0,05}(1)$. $n \geq 21$ uchun $x_n \in U_{0,05}(1)$ ya'ni $|x_n - 1| < 0{,}05$. Endi biz $\forall \varepsilon > 0$ son uchun $\exists n_0 \in \mathbb{N}$ ko'rsatishimiz kerakki shu sondan boshlab $|x_n - x_0| < \varepsilon$ bo'lsin (ya'ni $(1-\varepsilon; 1+\varepsilon)$ atrofida yotsin). Buning $|x_n - x_0| < \varepsilon$ tengsizlikni yechamiz.

$$|x_n - x_0| = \left|\frac{(-1)^n + n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}.$$

$\frac{1}{\varepsilon}$ har doim ham natural son bo'lmagani uchun bu sondan kattaroq $\left[\frac{1}{\varepsilon}\right]+1$ sonni $n_0$ sifatida tanlaymi, ya'ni $n_0=\left[\frac{1}{\varepsilon}\right]+1$. Demak $n_0=\left[\frac{1}{\varepsilon}\right]+1$ sondan boshlab $\left|x_n-x_0\right|<\varepsilon$ tengsizlik bajariladi, bu yerda $n>\left[\frac{1}{\varepsilon}\right]+1$.

**$x_n$ ketma-ketlikning limiti berilgan $a$ soni ekanini ta'rif yordamida isbotlang.**

**Namuna:**

**1-misol**. $\lim\limits_{n\to\infty}\frac{n}{n+10}=1$ tenglikni ta'rif yordimada isbotlang.

Agar $\lim\limits_{n\to\infty}x_n=a$ bo'lsa limit ta'rifiga ko'ra $\forall\varepsilon>0,\exists n_0\in\mathbb{N},\forall n\geq n_0:|x_n-a|<\varepsilon$ bo'lishi kerak. Demak biz $\forall\varepsilon>0$ son uchun $\exists n_0\in\mathbb{N}$ ko'rsatishimiz kerakki shu sondan boshlab $|x_n-a|<\varepsilon$ bo'lsin.

$$\left|\frac{n}{n+10}-1\right|<\varepsilon$$

$$\left|\frac{10}{n+1}\right|<\varepsilon\Rightarrow\frac{10}{n+10}<\varepsilon, n+10>\frac{10}{\varepsilon}, n>\frac{10}{\varepsilon}-10\Rightarrow n_0=\left[\frac{10}{\varepsilon}-10\right]+1=\left[\frac{10}{\varepsilon}-9\right]$$

$n_0=\left[\frac{10}{\varepsilon}-9\right]$ bo'lib ixtiyoriy $\varepsilon>0$ uchun $n_0$ ko'rsatildi.

**2-misol**. $x_n=\operatorname{arctg}n,\ a=\frac{\pi}{2}$ bo'lishini ta'rif yordimada isbotlang.

Ketma-ketlik limiti ta'rifiga ko'ra:

$$\forall\varepsilon>0,\exists n_0(\varepsilon),\forall n\geq n_0(\varepsilon):\left|x_n-a\right|<\varepsilon\left(x_n\in(a-\varepsilon,a+\varepsilon)\right).$$

Demak biz berilgan $\varepsilon>0$ son uchun shunday $n_0=n_0(\varepsilon)$ topishimiz kerakki $\forall n\geq n_0$ bo'lganda $x_n\in(a-\varepsilon,a+\varepsilon)$ ya'ni $\left|x_n-a\right|<\varepsilon$ bo'lsin.

$$\left|\operatorname{arctg}n-\frac{\pi}{2}\right|=\left|\frac{\pi}{2}-\operatorname{arctg}n\right|=\left|\operatorname{arcctg}n\right|=\left|\operatorname{arctg}\frac{1}{n}\right|=\arctan\frac{1}{n}<\varepsilon$$

$$\frac{1}{n}<tg\varepsilon\Rightarrow n>\frac{1}{tg\varepsilon}\Rightarrow n_0=\left[\frac{1}{tg\varepsilon}\right]+1\ .$$

**Masalalar**

1) $x_n=\dfrac{n}{n+1},\ a=1;$ 2) $x_n=\sqrt[n]{2},\ a=1;$ 3) $x_n=\dfrac{3n+1}{2n+1},\ a=\dfrac{3}{2};$

4) $x_n=\dfrac{\sin n}{n},\ a=0.$ 5) $x_n=arctgn,\ a=\dfrac{\pi}{2};$ 6) $x_n=\dfrac{(-1)^n}{n+3},\ a=0.$

$x_n$ **ketma-ketlikning limitga ega emasligini isbotlang.**

Na'muna:

$$x_n=\frac{n}{n+1}\sin\frac{\pi n}{2}.$$

Avval $\sin\dfrac{\pi n}{2}$ ifodaning qiymatlarini ko'ramiz:

$n=1\Rightarrow\sin\dfrac{\pi n}{2}=\sin\dfrac{\pi}{2}=1;$ $n=2\Rightarrow\sin\dfrac{\pi n}{2}=\sin\pi=0;$

$n=3\Rightarrow\sin\dfrac{\pi n}{2}=\sin\dfrac{3\pi}{2}=-1;$ $n=4\Rightarrow\sin\dfrac{\pi n}{2}=\sin 2\pi=0;$

$n=5\Rightarrow\sin\dfrac{\pi n}{2}=\sin\dfrac{5\pi}{2}=1.$

Demak qiymatlari har 4 ta qadamda takrorlanadi.

$n=4k+1$ bo'lgandagi qiymiy ketma-ketlik $\{x_{4k+1}\}\subset\{x_n\}$ limitini topamiz:

$$\lim_{n\to\infty}x_n=\lim_{k\to\infty}x_{4k+1}=\lim_{k\to\infty}\frac{4k+1}{4k+2}\sin\frac{(4k+1)\pi}{2}=\lim_{k\to\infty}\frac{1+\frac{1}{4k}}{1+\frac{1}{2k}}\sin\left(2\pi k+\frac{\pi}{2}\right)=1.$$

$n=4k+2$ bo'lgandagi qiymiy ketma-ketlik $\{x_{4k+2}\}\subset\{x_n\}$ limitini topsak:

$$\lim_{n\to\infty}x_n=\lim_{k\to\infty}x_{4k+2}=\lim_{k\to\infty}\frac{4k+2}{4k+3}\sin\frac{(4k+2)\pi}{2}=\lim_{k\to\infty}\frac{1+\frac{1}{4k}}{1+\frac{3}{4k}}\sin\left(2\pi k+\pi\right)=0.$$

Ketma-ketlik yaqinlashuvchi bo'lsa va $x_0$ songa yaqinlashsa uning qismiy ketma-ketliklari ham yaqinlashuvchi bo'ladi hamda $x_0$ soniga yaqinlashadi. Qismiy ketma-kema-ketliklari har xil songa yaqinlashgani uchun ketma-ketlik yaqinlashuvchi emas. Ya'ni $\lim\limits_{n\to\infty} x_n - \nexists$.

**Masalalar**.

1. $x_n = \sin\dfrac{n\pi}{2}$;

2. $x_n = \dfrac{(-1)^n n+1}{n+2}$;

3. $x_n = \dfrac{n}{n+1}\cos\pi n$;

4. $x_n = \dfrac{n-1}{n+1}\cos\dfrac{\pi n}{2}$.

Endi ketma-ketlik limitini topish qoidalarini keltirmiz. $\lim\limits_{n\to\infty} x_n$ va $\lim\limits_{n\to\infty} y_n$ limitlar mavjud bo'lsin.

1. $\lim\limits_{n\to\infty}\left(x_n \pm y_n\right) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;

2. $\lim\limits_{n\to\infty} x_n y_n = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$;

3. $\lim\limits_{n\to\infty} y_n \neq 0 \Rightarrow \lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$;

4. $x_n \le y_n \le z_n$ va $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a$ bo'lsa u holda $\lim\limits_{n\to\infty} y_n - \exists$ va $\lim\limits_{n\to\infty} y_n = a$;

5. $a > 1 \Rightarrow \lim\limits_{n\to\infty}\dfrac{n^k}{a^n} = 0$; $a > 1 \Rightarrow \lim\limits_{n\to\infty}\dfrac{\ln^k n}{n^a} = 0$; $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$;

6. (Chezaro-Shtols) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = \infty$ va $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} - \exists$ bo'lsa $\lim\limits_{n\to\infty}\dfrac{x_n - x_{n-1}}{y_n - y_{n-1}} - \exists$ va quyidagi tenglik o'rinli:

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}.$$

**Ketma-ketlik limitini toping.**

Na'muna:

$$1)\lim_{n\to\infty}\frac{3n^2-5n+4}{n^2-6n+7}=\lim_{n\to\infty}\frac{\frac{3n^2-5n+4}{n^2}}{\frac{n^2-6n+7}{n^2}}=\lim_{n\to\infty}\frac{3-\frac{5}{n}+\frac{4}{n^2}}{1-\frac{6}{n}+\frac{7}{n^2}}=\frac{3-0+0}{1-0+0}=3;$$

$$2)\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-\sqrt{n^2-n-1}\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+n+1}-\sqrt{n^2-n-1}\right)\left(\sqrt{n^2+n+1}+\sqrt{n^2-n-1}\right)}{\sqrt{n^2+n+1}+\sqrt{n^2-n-1}}=$$

$$=\lim_{n\to\infty}\frac{\left(n^2+n+1\right)-\left(n^2-n+1\right)}{\sqrt{n^2+n+1}+\sqrt{n^2-n-1}}=\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+n+1}+\sqrt{n^2-n-1}}=\lim_{n\to\infty}\frac{2}{\frac{\sqrt{n^2+n+1}}{n}+\frac{\sqrt{n^2-n-1}}{n}}=$$

$$=\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n}+\frac{1}{n^2}}}=\frac{2}{\sqrt{1+0+0}+\sqrt{1-0+0}}=1;$$

$$3)\lim_{n\to\infty}\left(\frac{2n+3}{2n+1}\right)^n=\lim_{n\to\infty}\left(\frac{1+\frac{3}{2n}}{1+\frac{1}{2n}}\right)^n=\lim_{n\to\infty}\frac{\left(1+\frac{3}{2n}\right)^n}{\left(1+\frac{1}{2n}\right)^n}=\lim_{n\to\infty}\frac{\left(\left(1+\frac{3}{2n}\right)^{\frac{2n}{3}}\right)^{\frac{3}{2}}}{\left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}}}=\frac{e^{\frac{3}{2}}}{e^{\frac{1}{2}}}=e;$$

$$4)\lim_{n\to\infty}\left(\sqrt{5}\sqrt[4]{5}\sqrt[8]{5}\cdot\ldots\cdot\sqrt[2^n]{5}\right)=\lim_{n\to\infty}\left(5^{\frac{1}{2}}5^{\frac{1}{4}}5^{\frac{1}{8}}\ldots5^{\frac{1}{2^n}}\right)=\lim_{n\to\infty}\left(5^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}}\right)=\lim_{n\to\infty}\left(5^{1-\frac{1}{2^n}}\right)=5^{1-0}=5;$$

$$5)\lim_{n\to\infty}\sqrt[n]{5^n+4^n}=\lim_{n\to\infty}\sqrt[n]{5^n\left(1+\left(\frac{4}{5}\right)^n\right)}=\lim_{n\to\infty}5\left(1+\left(\frac{4}{5}\right)^n\right)^{\frac{1}{n}}=5\left(1+0\right)^0=5.$$

Misollar.

$$1)\lim_{n\to\infty}\frac{2n+3}{3n+1};\qquad 2)\lim_{n\to\infty}\frac{3n^2-5n+4}{n^2-6n+7};\qquad 3)\lim_{n\to\infty}\frac{n^3-5n+1}{n^4+n+1};$$

$$4)\lim_{n\to\infty}\frac{3n^3-6n+7}{n^2+n};\qquad 5)\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right);\qquad 6)\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-\sqrt{n^2-n-1}\right);$$

$$7)\lim_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n(n+1)}\right);\qquad 8)\lim_{n\to\infty}\left(\sqrt{5}\sqrt[4]{5}\sqrt[8]{5}\cdot\ldots\cdot\sqrt[2^n]{5}\right);$$

$$9)\lim_{n\to\infty}\frac{5\sqrt{n^2+7}}{\sqrt[10]{n^{10}+3}};\qquad 10)\lim_{n\to\infty} n\left(\sqrt{3+\frac{n+1}{n}}-2\right);\qquad 11)\lim_{n\to\infty}\left(\frac{2n+3}{2n+1}\right)^n;$$

$$12)\lim_{n\to\infty}\sqrt[n]{5^n+4^n};\qquad 13)\lim_{n\to\infty}\sin^2\left(\pi\sqrt{n^2+n}\right);\qquad 14)\lim_{n\to\infty}\frac{(2n+1)^{50}}{(3n+1)^{20}(n+1)^{30}}.$$

## Yuqori va quyi limit

Yuqori va quyi limitlar mos ravishda quyidagicha aniqlanadi:

$$\overline{\lim_{n\to\infty}}\, x_n := \lim_{n\to\infty}\sup\{x_n, x_{n+1}, x_{n+2}\},\qquad \underline{\lim_{n\to\infty}}\, x_n := \lim_{n\to\infty}\inf\{x_n, x_{n+1}, x_{n+2}\}.$$

Quyidagicha aniqlash ham mumkin:

$$\overline{\lim_{n\to\infty}}\, x_n := \sup\left\{a:\exists\left\{x_{n_k}\right\}\subset\{x_n\}, x_{n_k}\to a\right\}, \underline{\lim_{n\to\infty}}\, x_n := \inf\left\{a:\exists\left\{x_{n_k}\right\}\subset\{x_n\}, x_{n_k}\to a\right\}.$$

Namuna:

Quyidagi ketma-ketlikni yuqori va quyi limitini toping:

$$a_n=\frac{n+1}{2n+1}\sin\frac{\pi n}{4}.$$

Yechish: $\sin\frac{\pi n}{4}$ ketma-ketlikni qiymatlar to’plami $\left\{0;\pm\frac{1}{\sqrt{2}};\pm 1\right\}$ bo’lib, har 8 ta qadamda qiymatlari takrorlanadi. $\sin\frac{\pi n}{4}=1$ qiymatga $n=8k+2$ va $\sin\frac{\pi n}{4}=-1$ qiymatga $n=8k+6$ bo’lganda erishadi. Shuning uchun

$$\overline{\lim_{n\to\infty}} x_n = \lim_{k\to\infty} x_{8k+2} = \lim_{k\to\infty}\frac{8k+3}{16k+5} = \frac{1}{2}, \quad \underline{\lim_{n\to\infty}} x_n = \lim_{k\to\infty} x_{8k+6} = \lim_{k\to\infty}\left(-\frac{8k+7}{16k+13}\right) = -\frac{1}{2}.$$

$\overline{\lim\limits_{n\to\infty}} x_n$ **va** $\underline{\lim\limits_{n\to\infty}} x_n$ **limitlarni toping.**

**Masalalar**

Yuqori va quyi limitlarini toping.

1. $x_n = \dfrac{(-1)^n n+1}{n+2}$; 2. $x_n = \sin\dfrac{n\pi}{2}$; 3. $x_n = \dfrac{n}{n+1}\cos\pi n$; 4. $x_n = \dfrac{n-1}{n+1}\cos\dfrac{\pi n}{2}$.

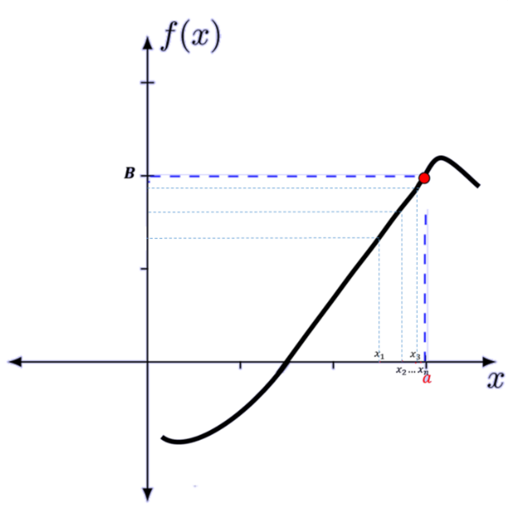
## Funksiya limiti

$f(x) = \dfrac{\sqrt{1+3x}-1}{x}$ funksiya $x=0$ nuqtada aniqlanmagan. Biroq $x$ ning nolga yaqin bo'lgan nuqtalarda qiymatlarini o'zgarishini ko'ramiz.

| $x$ | 1 | 0.63 | 0.52 | 0.32 | 0.23 | 0.1 | 0.07 | 0.003 | 0.001 | 0.0001 | ... | $x\to 0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 1.111... | 1.153... | 1.25... | 1.304... | 1.401... | 1.428... | 1.496... | 1.498... | 1.499... | ... | 1.5 |
| $x_n$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | ... | $x_n\to 0$ |

$x_n \to 0$ bo'lganda $f(x_n)$ funksiya qiymatlari 1.5 ga yaqinlashyapdi.

**Ta'rif**. $f(x)$ – funksiya $x_0$ nuqtaning biror atrofida aniqlangan bo'lsin. Agar $\forall\{x_n\}, x_n \neq x_0 : x_n \to x_0$ ketma-ketlik olganda ham $f(x_n) \to B$ bo'lsa u holda $B$ soni $f(x)$ funksiyaning $x \to x_0$ bo'lgandagi limiti deyiladi va $\lim\limits_{x\to x_0} f(x) = B$ kabi yoziladi:



$\lim\limits_{x\to 0}\frac{\sqrt{1+3x}-1}{x}$ limitni $x_n \to 0$ bo'luvchi bir nechta ketma-ketliklar misolida ko'ramiz.

$x_n = \frac{2}{n}+\frac{3}{n^2}\to 0 \Rightarrow \frac{3}{2}$

$$f(x_n)=\frac{\sqrt{1+3\left(\frac{2}{n}+\frac{3}{n^2}\right)}-1}{\frac{2}{n}+\frac{3}{n^2}}=\frac{\sqrt{1+\frac{6}{n}+\frac{9}{n^2}}-1}{\frac{2}{n}+\frac{3}{n^2}}=\frac{1+\frac{3}{n}-1}{\frac{2}{n}+\frac{3}{n^2}}=\frac{3}{2+\frac{3}{n}}\to\frac{3}{2};$$

$x_n = \frac{1}{3}\left(e^{\frac{2}{n}}-1\right)\to 0 \Rightarrow$

$$f(x_n)=\frac{\sqrt{1+\frac{1}{3}\left(e^{\frac{2}{n}}-1\right)}-1}{\frac{1}{3}\left(e^{\frac{2}{n}}-1\right)}=3\frac{e^{\frac{1}{n}}-1}{e^{\frac{2}{n}}-1}=3\frac{e^{\frac{1}{n}}-1}{\left(e^{\frac{1}{n}}-1\right)\left(e^{\frac{1}{n}}+1\right)}=\frac{3}{e^{\frac{1}{n}}+1}\to\frac{3}{2};$$

$x_n = \mathrm{arcctg}\, n \to 0 \Rightarrow$

$$f(x_n)=\frac{\sqrt{1+3\,\mathrm{arcctg}\, n}-1}{\mathrm{arcctg}\, n}=\frac{\left(\sqrt{1+3\,\mathrm{arcctg}\, n}-1\right)\left(\sqrt{1+3\,\mathrm{arcctg}\, n}+1\right)}{\left(\sqrt{1+3\,\mathrm{arcctg}\, n}+1\right)\mathrm{arcctg}\, n}=$$

$$=\frac{3\,\mathrm{arcctg}\, n}{\left(\sqrt{1+3\,\mathrm{arcctg}\, n}+1\right)\mathrm{arcctg}\, n}=\frac{3}{\sqrt{1+3\,\mathrm{arcctg}\, n}+1}\to\frac{3}{\sqrt{1+3\cdot 0}+1}=\frac{3}{2}.$$

Umuman olganda $\forall\{x_n\}\to 0, x_n\neq 0$ ketma-ketlik uchun

$$f(x_n)=\frac{\sqrt{1+3x_n}-1}{x_n}=\frac{\left(\sqrt{1+3x_n}-1\right)\left(\sqrt{1+3x_n}+1\right)}{x_n\left(\sqrt{1+3x_n}+1\right)}=\frac{3}{\sqrt{1+3x_n}+1}\to\frac{3}{2}$$

o'rinli. Demak

$$\lim_{x\to 0}\frac{\sqrt{1+3x}-1}{x}=\frac{3}{2}.$$

**Quyidagi funksiyalar limitga ega emasligini isbotlang**.

Na'muna:

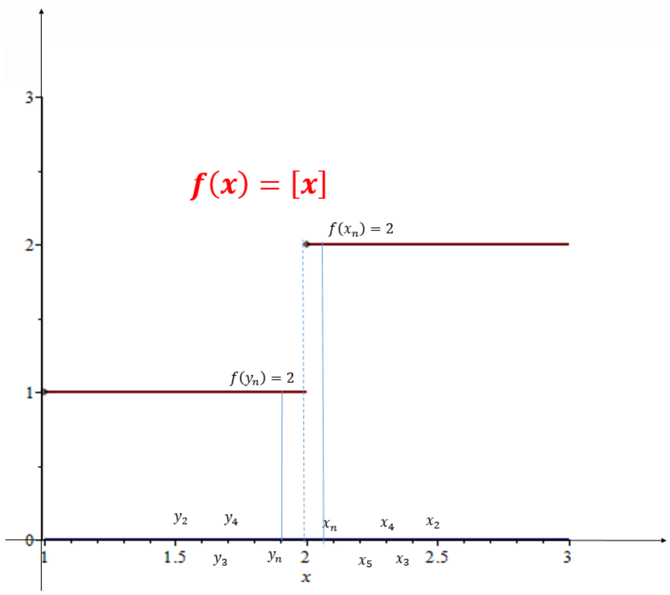
**1-misol.** $f(x)=[x]$ funksiya $x\to 2$ bo'ganda limitga ega emasligini isbotlang.

Yechish: $x_n = 2+\frac{1}{n}\to\infty$ ketma-ketlik olamiz. $n\geq 2$ uchun $2<x_n=2+\frac{1}{n}<3$ tengsizlik bajarilgani uchun $[x_n]=2$ va $n\to\infty$ bo'lganda $[x_n]\to 2$ ya'ni $f(x_n)\to 2$

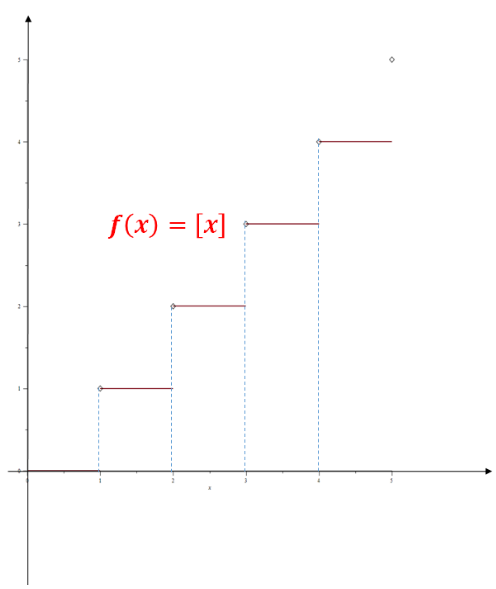
bo'ladi. Agar $y_n = 2 - \frac{1}{n} \to 2$ boshqa ketma-ketlik olsak $n \geq 2$ uchun $1 < y_n < 2$ bo'lib $[y_n] \to 1$ ya'ni $f(y_n) \to 1$.

$$x_n = 2 + \frac{1}{n} \to 2 \Rightarrow f(x_n) \to 2;$$

$$y_n = 2 - \frac{1}{n} \to 2 \Rightarrow f(y_n) \to 1.$$




Ammo Geyni ta'rifiga ko'ra $\forall\{x_n\} \to 2$ ketma-ketlik olganda ham $f(x_n)$ bitta songa intilishi kerak edi. Shuning uchun $\lim\limits_{x\to 2}[x]$ mavjud emas.



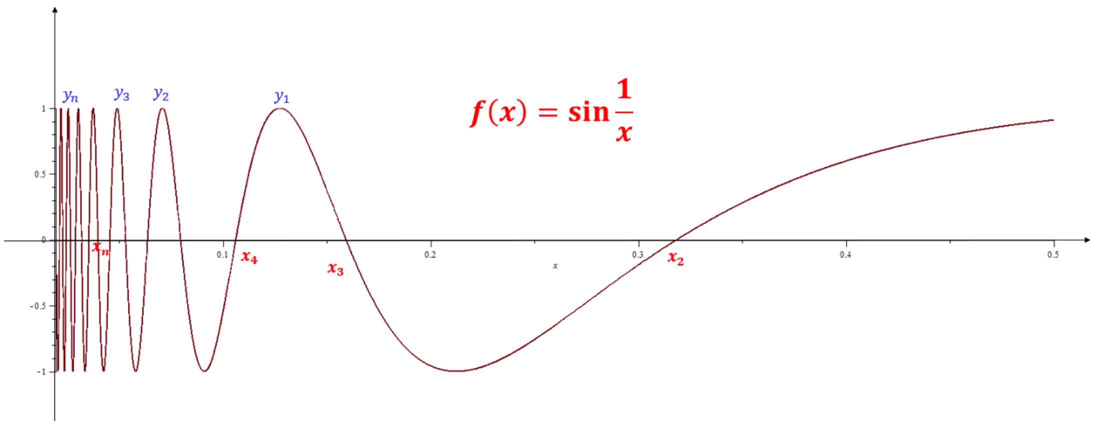

Umumiy holda $m \in Z$ butun son bo'lganda $\lim\limits_{x\to m}[x]$ mavjud emas.

**2-misol.** $f(x) = \sin\frac{1}{x}$ funksiya $x \to 0$ bo'ganda limitga ega emasligini isbotlang.

Yechilishi. Nolga inltiluvchi noldan farqli ikkita farqli ketma-ketlik olamiz.

$$x_n = \frac{1}{\pi n} \to 0 \Rightarrow f(x_n) = \sin \pi n = 0 \to 0;$$

$$y_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \to 0 \Rightarrow f(y_n) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = \sin\frac{\pi}{2} = 1 \to 1.$$




$x_n \to 0$ bo'ladigan ikkita turli ketma-ketlik uchun $f(x_n)$ ikki xil songa intiladi. Shuning uchun $\lim\limits_{x\to 0} \sin\frac{1}{x}$ mavjud emas.

**Misollar**

Funksiya $x \to a$ bo'lganda limitga ega emasligini isbotlang.

1. $f(x) = \cos\frac{1}{x}, x \to 0;$

2. $f(x) = sgn(x) = \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}, x \to 0;$

3. $f(x) = \sin\ln x, x \to 0;$

4. $f(x) = \begin{cases} 2x+1, x < 1 \\ 5x-1, x \geq 1 \end{cases}, x \to 1.$

## Limitni hisoblash qoidalari

$\lim\limits_{x\to x_0} f(x), \lim\limits_{x\to x_0} g(x) -$ limitlar mavjud bo'lsin.

1. $\lim\limits_{x\to x_0}\left(f(x) \pm g(x)\right) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x);$

2. $\lim\limits_{x\to x_0} f(x)g(x) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x);$

3. $\lim\limits_{x\to x_0} g(x)\neq 0 \Rightarrow \lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$;

4. $\lim\limits_{x\to 0} x=0;$

5. Ajoyib limitlar: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ , $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$;

6. $\lim\limits_{x\to 0}\dfrac{(1+x)^a-1}{x}=a, \qquad a>0\Rightarrow \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$ ;

7. $\lim\limits_{x\to 0}\dfrac{tgx}{x}=1,\ \lim\limits_{x\to 0}\dfrac{\arcsin x}{x}=1,\ \lim\limits_{x\to 0}\dfrac{arctgx}{x}=1;$

8. $\lim\limits_{x\to a} f(x)-\exists \Leftrightarrow \lim\limits_{x\to a+} f(x)=\lim\limits_{x\to a-} f(x);$

9. $f(x)\le g(x)\le h(x), \forall x\in(a-\varepsilon, a+\varepsilon)$ va $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} h(x)=B$ bo’lsa u holda $\lim\limits_{x\to a} g(x)-\exists$ va $\lim\limits_{x\to a} g(x)=B$.

**Limitlarni hisoblang**.

Na’muna:

**1-misol**. $\lim\limits_{x\to 0}\dfrac{(1+x)^4-1-4x}{x^2}$;

Yechish.

Nyuton binomi formulasiga ko’ra

$(1+x)^4-1-4x=1+4x+6x^2+4x^3+x^4-1-4x=x^2\left(6+4x+x^2\right)$ .

Demak

$\lim\limits_{x\to 0}\dfrac{(1+x)^4-1-4x}{x^2}=\lim\limits_{x\to 0}\left(6+4x+x^2\right)=6.$

**2-misol**. $\lim\limits_{x\to\infty}\dfrac{\ln\left(x^3+x-1\right)}{\ln\left(x^2-3x+10\right)}$;

Yechish:

$$\lim_{x\to\infty}\frac{\ln\left(x^3+x-1\right)}{\ln\left(x^2-3x+10\right)}=\lim_{x\to\infty}\frac{\ln\left(x^3\left(1+\frac{1}{x^2}-\frac{1}{x^3}\right)\right)}{\ln\left(x^2\left(1-\frac{3}{x}+\frac{10}{x^2}\right)\right)}=\lim_{x\to\infty}\frac{\ln x^3+\ln\left(1+\frac{1}{x^2}-\frac{1}{x^3}\right)}{\ln x^2+\ln\left(1-\frac{3}{x}+\frac{10}{x^2}\right)}=$$

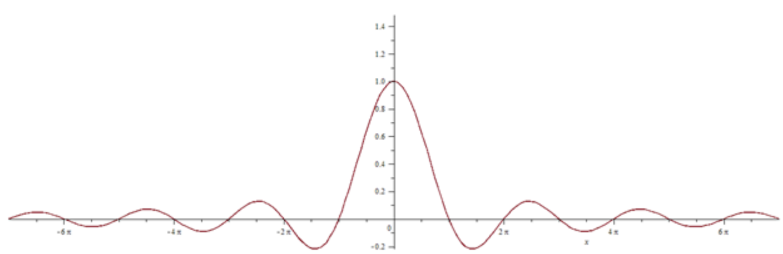
$$=\lim_{x\to\infty}\frac{3\ln x+\ln\left(1+\frac{1}{x^2}-\frac{1}{x^3}\right)}{2\ln x+\ln\left(1-\frac{3}{x}+\frac{10}{x^2}\right)}=\lim_{x\to\infty}\frac{3+\dfrac{\ln\left(1+\frac{1}{x^2}-\frac{1}{x^3}\right)}{\ln x}}{2+\dfrac{\ln\left(1-\frac{3}{x}+\frac{10}{x^2}\right)}{\ln x}}=\frac{3}{2}.$$

Misollar.

1. $\lim\limits_{x\to1}\frac{x^3-3x+2}{x^2-3x+2}$;
2. $\lim\limits_{x\to1}\frac{x-x^3}{2x+x^2-3}$;
3. $\lim\limits_{x\to0}\frac{(1+x)^5-1-5x}{x^2}$;
4. $\lim\limits_{x\to7}\frac{\sqrt{x+2}-3}{x^2-8x+7}$;

**Ajoyib limitlardan foydalanib limitlarni hisoblang**.

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

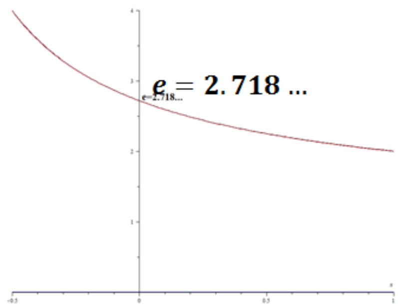


Misollar.

1. $\lim\limits_{x\to0}\frac{\sin 5x}{4x}$;
2. $\lim\limits_{x\to0}\frac{\sin 3x}{\sin 5x}$;

3. $\lim\limits_{x\to 0}\frac{\cos 3x-\cos 2x}{x^2}$;
4. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$.

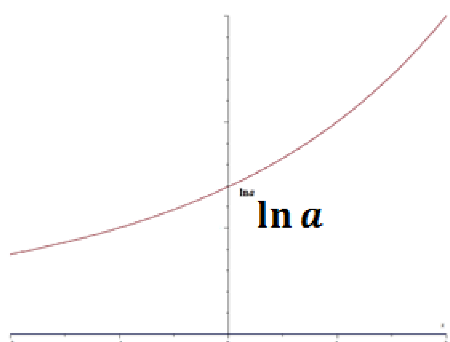
$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$




Misollar.

5. $\lim\limits_{x\to 0}(1+3x)^{\frac{5}{x}}$;
6. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}$;
7. $\lim\limits_{x\to 0}\frac{\ln(1+2x)}{\sin 3x}$;
8. $\lim\limits_{x\to 0}\ln\frac{1+5x}{\operatorname{tg}2x+\ln(1+x)}$.

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a, a>0$$




Misollar.

9. $\lim\limits_{x\to 0}\frac{3^x-2^x}{x}$;
10. $\lim\limits_{x\to 1}\frac{5^x-5}{x-1}$, [hint: $x-1=t$];
11. $\lim\limits_{x\to 0}\frac{3^x-\cos x}{5x}$;

12. $\lim\limits_{x\to 0}\frac{e^x-1}{\ln(1+2x)}$.

$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a$$

**Misollar**

13. $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$;
14. $\lim\limits_{x\to 1}\frac{x^n-1}{x^m-1}$;
15. $\lim\limits_{x\to 0}\frac{2^x-\sqrt[3]{x+1}}{x}$;
16. $\lim\limits_{x\to 0}\frac{3^x-\sqrt{1+2x}}{\sin 5x+\ln(1+x)}$.

Namuna:

1) $$\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}=\left[t=x-7\to 0, x=t+7\right]=\lim_{t\to 0}\frac{\sqrt{t+9}-\sqrt[3]{t+27}}{\sqrt[4]{16+t}-2}=$$

$$\lim_{t\to 0}\frac{3\left(\sqrt{1+\frac{t}{9}}-\sqrt[3]{1+\frac{t}{27}}\right)}{2\left(\sqrt[4]{1+\frac{t}{16}}-1\right)}=\frac{3}{2}\lim_{t\to 0}\frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}}-\left(1+\frac{t}{27}\right)^{\frac{1}{3}}}{\left(1+\frac{t}{16}\right)^{\frac{1}{4}}}=\frac{3}{2}\lim_{t\to 0}\frac{\frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}}-1}{t}-\frac{\left(1+\frac{t}{27}\right)^{\frac{1}{3}}-1}{t}}{\frac{\left(1+\frac{t}{16}\right)^{\frac{1}{4}}-1}{t}}=$$

$$=\frac{3}{2}\lim_{t\to 0}\frac{\frac{1}{9}\frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}}-1}{\frac{t}{9}}-\frac{1}{27}\frac{\left(1+\frac{t}{27}\right)^{\frac{1}{3}}-1}{\frac{t}{27}}}{\frac{1}{16}\frac{\left(1+\frac{t}{16}\right)^{\frac{1}{4}}-1}{\frac{t}{16}}}=\frac{3}{2}\frac{\frac{1}{9}\cdot\frac{1}{2}-\frac{1}{27}\cdot\frac{1}{3}}{\frac{1}{16}\cdot\frac{1}{4}}$$

2) $$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}=\lim_{x\to 0}\left(1+(\cos x-1)\right)^{\frac{1}{x^2}}=\lim_{x\to 0}\left(1+(\cos x-1)\right)^{\frac{1}{\cos x-1}\frac{\cos x-1}{x^2}}=$$

$$=\lim_{x\to 0}\left(1+(\cos x-1)\right)^{\frac{1}{\cos x-1}\frac{-2\sin^2\frac{x}{2}}{x^2}}=\lim_{x\to 0}\left(1+(\cos x-1)\right)^{-\frac{1}{2}\frac{1}{\cos x-1}\left(-\frac{1}{2}\right)\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}=e^{-\frac{1}{2}}$$

**Misollar**

17. $\lim\limits_{x\to4}\frac{\sqrt[3]{x+4}-2}{\sqrt{x}-2}$;

18. $\lim\limits_{x\to7}\frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$.

*Agar* $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=1$ *bo'lsa* $f$ *va* $g$ *funksiyalar* $x\to x_0$ *bo'lganda ekvivalent deyiladi va* $f(x)\sim g(x)$ *kabi belgilanadi. Funksiyaning o'rniga uning ekvivalentidan foydalanish limit hisoblashni ancha osonlashtiradi.*

$x\to 0$ **bo'lganda limitda quyidagi almashtirishlarni qilish mumkin:**

$$\sin x\sim x, a^x\sim 1+x\ln a,\ (1+x)^a\sim 1+ax,\ \cos x\sim 1-\frac{x^2}{2},\ 1+x\sim e^x$$

**Namuna:**

$$\lim_{x\to0}\frac{\left(3^x-\sqrt[3]{1+2x}\right)}{\sin 5x+\ln(1-4x)}=\left[\begin{array}{c}3^x\sim1+x\ln3\\ \sqrt[3]{1+2x}=(1+2x)^{\frac{1}{3}}\sim1+\frac{2x}{3}\\ \sin5x\sim5x\\ 1-4x\sim e^{-4x}\end{array}\right]=\lim_{x\to0}\frac{1+x\ln3-\left(1-\frac{2x}{3}\right)}{5x+\ln e^{-4x}}=$$

$$=\lim_{x\to0}\frac{x\left(\ln3+\frac{2}{3}\right)}{x}=\ln3+\frac{2}{3}.$$

**Misollar**

19. $\lim\limits_{x\to0}\frac{\sqrt[n]{1+ax}\sqrt[m]{1+bx}-1}{x}$;

20. $\lim\limits_{x\to0}\left(\frac{1+2x}{1+5x}\right)^{\frac{3}{x}}$;

21. $\lim\limits_{x\to0}(\cos x)^{\frac{1}{x^2}}$;

22. $\lim\limits_{x\to1}x^{\operatorname{tg}\frac{\pi x}{2}}$ [**Hint**: $x-1=t, \operatorname{tg}t\sim t$];

23. $\lim\limits_{x\to0}\frac{5^x-\sqrt[3]{1+3x}}{\ln(1+3x)+\operatorname{tg}\frac{x}{2}}$;

24. $\lim\limits_{x\to0}\frac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}$.

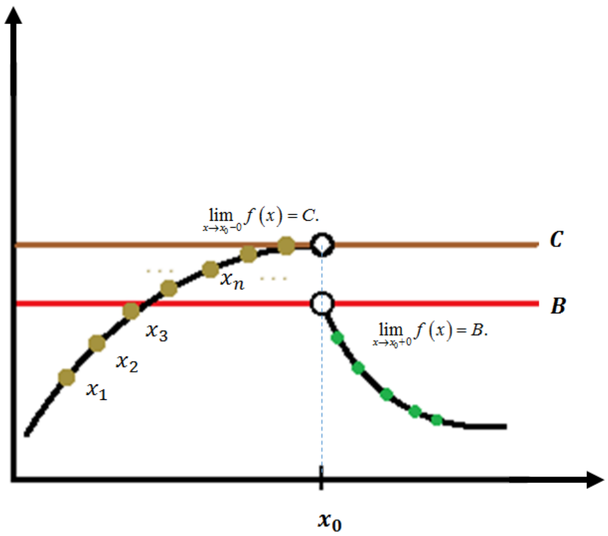
## O’ng va chap limitlar

**Ta’rif.** $f(x)$ – funksiya $x_0$ nuqtaning biror atrofida aniqlangan bo’lsin. Agar $\forall\{x_n\}, x_n > x_0 : x_n \to x_0$ ketma-ketlik olganda ham $f(x_n) \to B$ bo’lsa u holda $B$ soni $f(x)$ funksiyaning $x \to x_0 + 0$ bo’lgandagi o’ng limiti deyiladi va quyidagicha yoziladi:

$$\lim_{x\to x_0+0} f(x) = B.$$

Shuningdek agar $\forall\{x_n\}, x_n < x_0 : x_n \to x_0$ ketma-ketlik olganda ham $f(x_n) \to C$ bo’lsa u holda $B$ soni $f(x)$ funksiyaning $x \to x_0 - 0$ bo’lgandagi o’ng limiti deyiladi va quyidagicha yoziladi:

$$\lim_{x\to x_0-0} f(x) = C.$$




Namuna:

$f(x) = \ln\dfrac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}}$ uchun $\lim\limits_{x\to+\infty} f(x) - \lim\limits_{x\to-\infty} f(x)$ ning qiymatini toping.

$$\lim_{x\to+\infty}\ln\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}} = \lim_{x\to+\infty}\ln\frac{1+\sqrt{1+\dfrac{a^2}{x^2}}}{1+\sqrt{1+\dfrac{b^2}{x^2}}} = \ln\frac{1+\sqrt{1+0}}{1+\sqrt{1+0}} = 0;$$

$$\lim_{x\to-\infty}\ln\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}} = [x=-t, t\to\infty] = \lim_{t\to+\infty}\ln\frac{-t+\sqrt{t^2+a^2}}{-t+\sqrt{t^2+b^2}} = \lim_{t\to+\infty}\ln\frac{\sqrt{t^2+a^2}-t}{\sqrt{t^2+b^2}-t} =$$

$$= \lim_{t\to+\infty}\ln\frac{\left(\sqrt{t^2+a^2}-t\right)\left(\sqrt{t^2+a^2}+t\right)\left(\sqrt{t^2+b^2}+t\right)}{\left(\sqrt{t^2+b^2}-t\right)\left(\sqrt{t^2+b^2}+t\right)\left(\sqrt{t^2+a^2}-t\right)} = \lim_{t\to+\infty}\ln\frac{\left(t^2+a^2-t^2\right)\left(\sqrt{t^2+b^2}+t\right)}{\left(t^2+b^2-t^2\right)\left(\sqrt{t^2+a^2}+t\right)} =$$

$$= \lim_{t\to+\infty}\ln\frac{a^2\left(\sqrt{t^2+b^2}+t\right)}{b^2\left(\sqrt{t^2+a^2}+t\right)} = \lim_{t\to+\infty}\ln\frac{a^2\left(\sqrt{1+\dfrac{b^2}{t^2}}+1\right)}{b^2\left(\sqrt{1+\dfrac{a^2}{t^2}}+1\right)} = \ln\frac{a^2}{b^2};$$

Demak $\lim\limits_{x\to+\infty} f(x) - \lim\limits_{x\to-\infty} f(x) = 0 - \ln\dfrac{a^2}{b^2} = \ln\dfrac{b^2}{a^2}$.

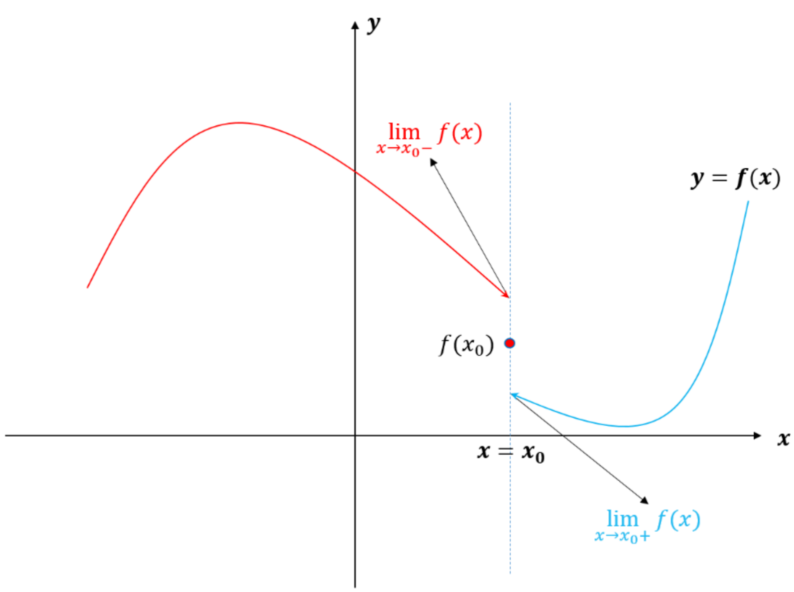
25. $\lim\limits_{x\to+\infty} \operatorname{arctg} x$, $\lim\limits_{x\to-\infty} \operatorname{arctg} x$;
26. $\lim\limits_{x\to 0-} \dfrac{\arccos(1-x)}{\sqrt{x}}$ ;
27. $\lim\limits_{x\to 1+} \operatorname{arctg}\frac{1}{x-1}$, $\lim\limits_{x\to 1-} \operatorname{arctg}\frac{1}{x-1}$;
28. $\lim\limits_{x\to 2+} [x] - \lim\limits_{x\to 2-} [x]$;
29. $f(x) = \ln\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}}$, $\lim\limits_{x\to+\infty} f(x) - \lim\limits_{x\to-\infty} f(x) =?$;
30. $\lim\limits_{x\to\infty} \sin^2 \pi\left(\sqrt{x+1} - \sqrt{x}\right)$.

## Funksiyaning uzluksizligi

*$f(x)$-funksiya $x \to x_0$ bo'lganda $f(x) \to f(x_0)$ bo'lsa ya'ni*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

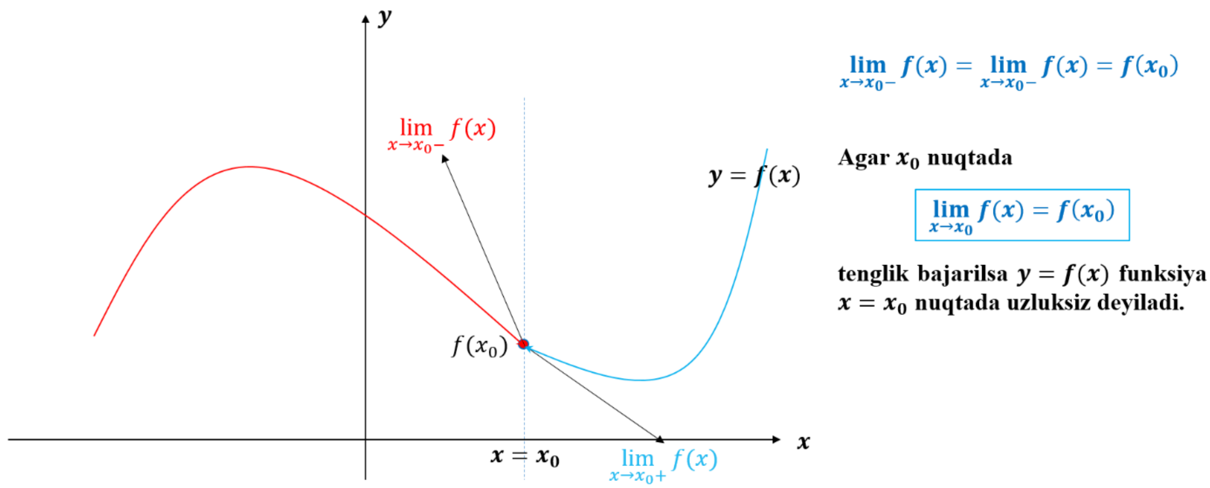
*u holda $f(x)$ funksiya $x = x_0$ nuqtada uzluksiz deyiladi.*




$f(x)$ funksiya $x = x_0$ nuqtada uzluksiz bo'lishi uchun quyidagi shartlar bajarilishi kerak:

1-shart. $\lim\limits_{x\to x_0} f(x)$ mavjud bo'lishi ( $\lim\limits_{x\to x_0-} f(x) = \lim\limits_{x\to x_0+} f(x)$ ya'ni o'ng va chap limitlar mavjud va teng bo'lishi);

2-shart. $b = \lim_{x \to x_0} f(x)$ limitini qiymati funksiyani $x = x_0$ nuqtadagi qiymatiga teng bo'lishi ya'ni $b = f(x_0)$.




**Funksiyani uzluksizlikka tekshiring.**

Na'muna.

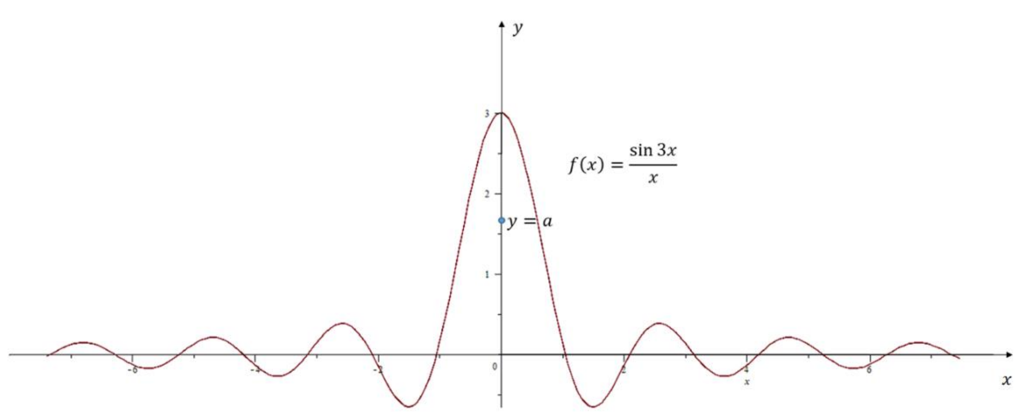
**1-misol.** $a$ ning qayday qiymatida $f(x)$ funksiya uzluksiz bo'ladi?

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, x \neq 0 \\ a, x = 0 \end{cases}$$

Yechish. $x \to 0$ bo'lganda

$$\lim_{x \to 0} \frac{\sin 3x}{x} = 3$$

va funksiya qiymati $f(0) = a$.



Funksiya uzluksiz bo'lishi uchun $x \to 0$ dagi limiti $x = 0$ dagi qiymatiga teng bo'lishi kerak, ya'ni $a = 3 - misol.$

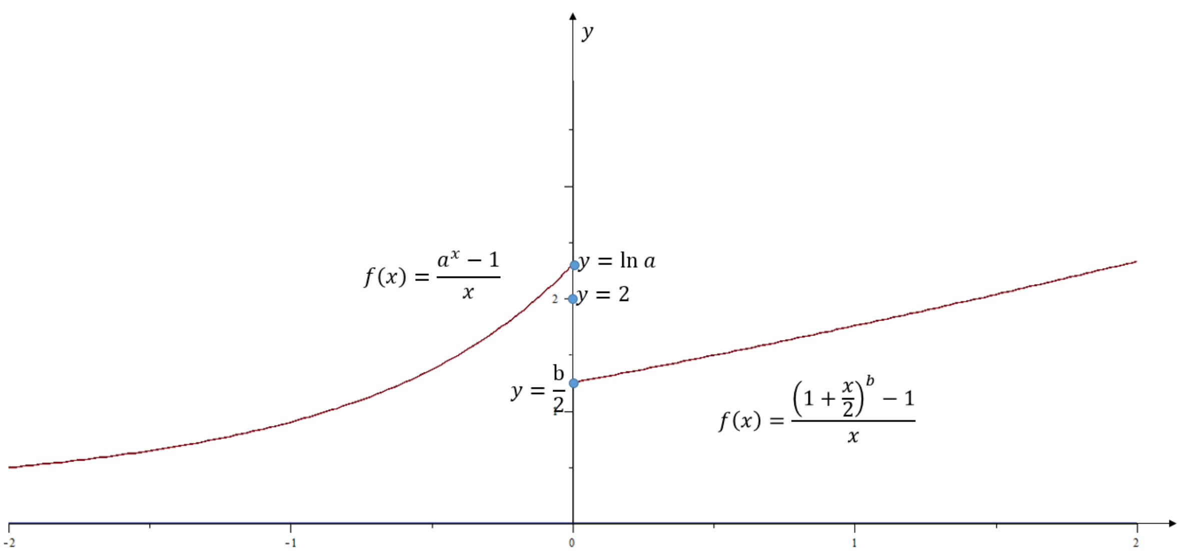
Javob: $a = 3 - misol.$

**2-misol.** $a$ va $b$ ning qayday qiymatida $f(x)$ funksiya uzluksiz bo'ladi?

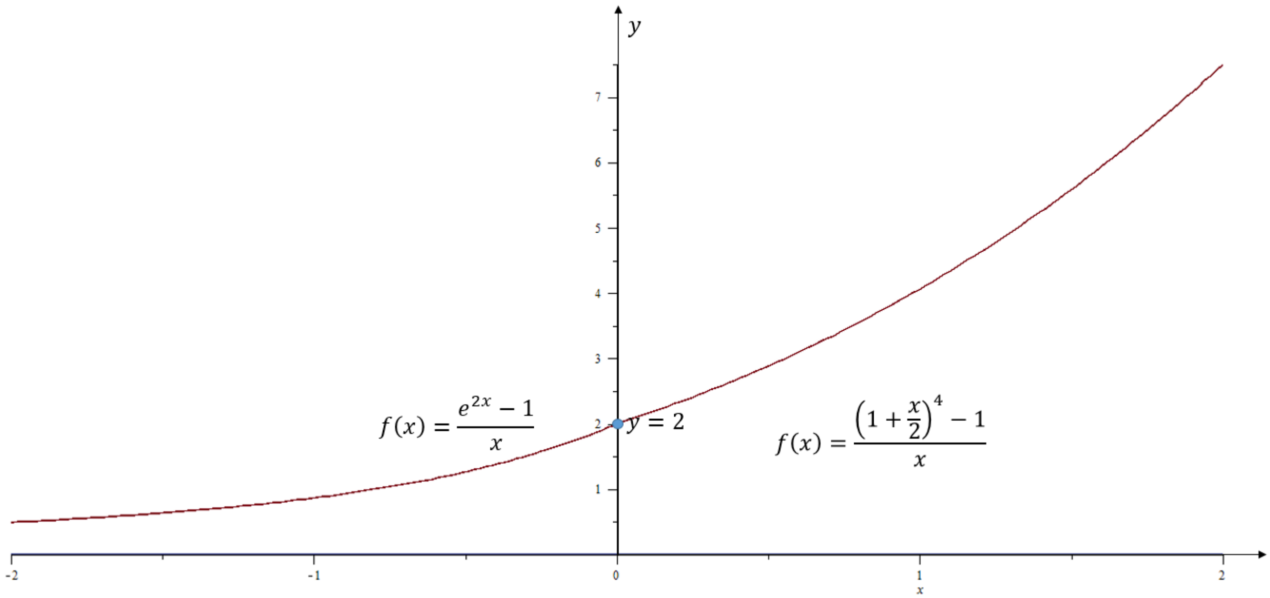
$$f(x) = \begin{cases} \dfrac{a^x - 1}{x}, x < 0 \\ 2, x = 0 \\ \dfrac{\left(1 + \frac{x}{2}\right)^b - 1}{x}, x > 0 \end{cases}$$

**Yechish**: Funksiya uzluksiz bo'lishi uchun u limitga ega bo'lishi kerak, ya'ni chap va o'ng limitlari teng bo'lishi kerak:

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{a^x - 1}{x} = \ln a \text{ , } \lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{\left(1 + \frac{x}{2}\right)^b - 1}{x} = \frac{b}{2}$$




Demak $\frac{b}{2} = \ln a$. 2-shartga ko'ra bu limit qiymati funksiya qiymatiga teng bo'lishi kerak ya'ni $\frac{b}{2} = \ln a = 2 - misol$. Shuning uchun $b = 4, a = e^2$.



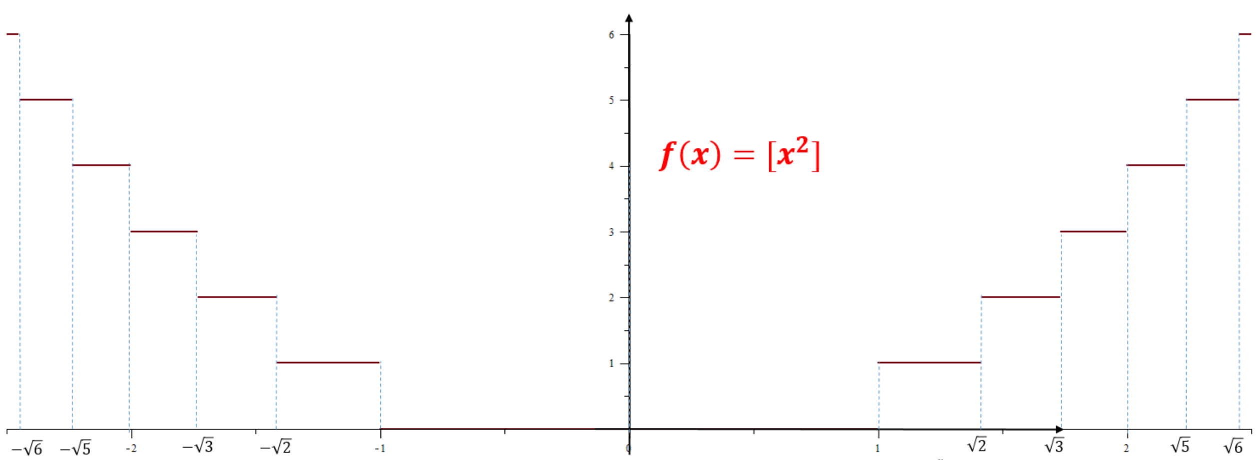

Javob: $b = 4, a = e^2$

**3-misol.** Funksiyaning uzulish nuqtalarini toping: $f(x) = [x^2]$.

Yechish. Funksiya uzluksiz bo'lishi uchun u limitga ega bo'lishi kerak. $g(x) = [x]$ funksiya $x = n \in Z$ nuqtalarda limitga ega emas. Ya'ni shu nuqtalarda uzilgan. Bundan $x^2 = n \in \mathbb{Z}$ bo'ladigan nuqtalarda (masalan $x = \sqrt{n}$ bo'ladigan nuqtalarda):

$$\lim_{x\to\sqrt{n}-}\left[x^2\right] = n-1, \lim_{x\to\sqrt{n}+}\left[x^2\right] = n \Rightarrow \lim_{x\to\sqrt{n}-}\left[x^2\right] \neq \lim_{x\to\sqrt{n}+}\left[x^2\right]$$

bo'lib $f(x) = [x^2]$ funksiya uzilgan.




Javob: $x = \pm\sqrt{n}$ nuqtalarda uzulishga ega.

Misollar.

1. $f(x)=\begin{cases} ax+2, x<1 \\ 3x-a, x\geq 1 \end{cases}$, $\quad f(x)=\operatorname{arctg} e^{\frac{1}{x+1}}$, $\quad f(x)=\begin{cases} \frac{2\sin x-1}{x-\frac{\pi}{6}}, x\neq \frac{\pi}{6} \\ a \qquad , x=\frac{\pi}{6} \end{cases}$

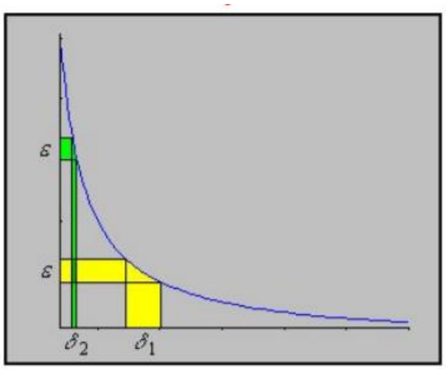
2. $f(x)=\begin{cases} \frac{(1+2x)^a-1}{x}, x<0 \\ 5, \qquad x=0 \\ \frac{\ln(1+bx)}{x}, x>0 \end{cases}$, $\quad f(x)=\begin{cases} \sin\frac{1}{x}, x\neq 0 \\ 0 \quad , x=0 \end{cases}$

3. $f(x)=\begin{cases} \frac{e^{bx}-e^{ax}}{x}, \qquad x<0 \\ 2a+b-1, \quad x=0 \\ \frac{\sqrt{1+ax}-\sqrt[3]{1+bx}}{x}, x>0 \end{cases}$ $\quad f(x)=\begin{cases} x, x\in Q \\ 0, x\notin Q \end{cases}$

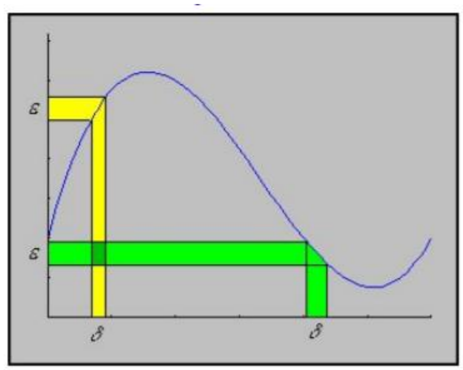
4. $f(x)=\left[\frac{1}{x}\right]$, $\quad f(x)=\cos\frac{1}{x-1}$

## Funksiyaning tekis uzluksizligi

*Agar ixtiyoriy $\varepsilon>0$ son olinganda ham shunday $\delta>0$ son topilib, $|x'-x''|<\delta$ va $x', x''\in X$ bo'lganda $|f(x')-f(x'')|<\varepsilon$ bo'lsa u holda $f(x)$ funksiya $X$ to'plamda tekis uzluksiz deyiladi.*




Tekis uzluksiz bo'lmagan funksiyaning grafigi

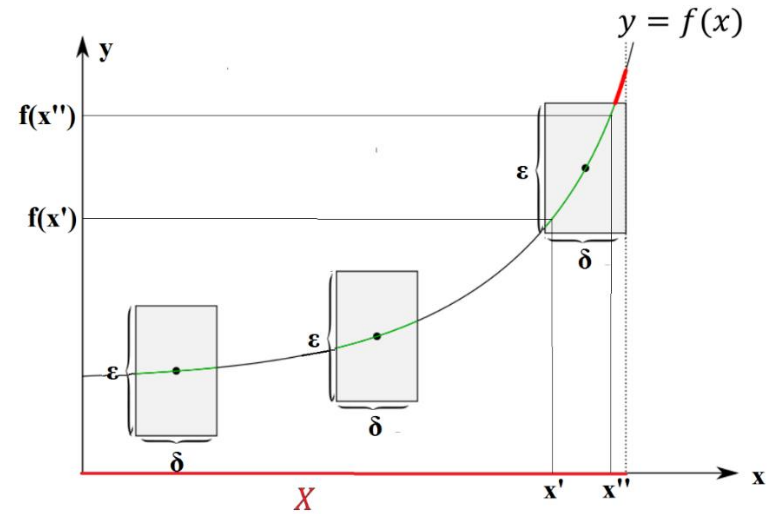



Tekis uzluksiz funksiyaning grafigi

Ya'ni

$$\forall\varepsilon>0, \exists\delta>0, |x'-x''|<\delta, \forall x', x''\in X : \left|f(x')-f(x'')\right|<\varepsilon$$

δ sonini ε bilan bog’liq qilib topish kerak. $|x' - x''| < \delta$ bo’lsin deb $\left|f(x') - f(x'')\right|$ ayirmani δ orqali bo’glash mumkin bo’lsa tekis uzluksiz bo’ladi.

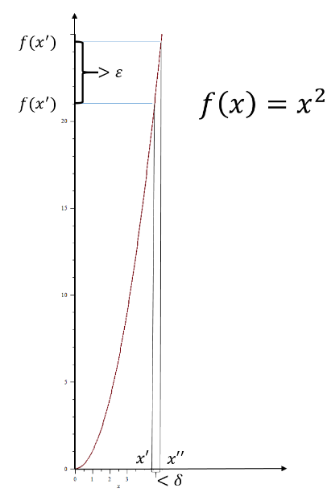



**1-misol**. $f(x) = x^2$ funksiyani $X = [0; 2]$ to’plamda tekis uzluksizlikka tekshiring.

**Yechish**: $|x' - x''| < \delta$ bo’lsin.

$$\left|f(x') - f(x'')\right| = \left|x'^2 - x''^2\right| = |x' - x''||x' + x''| < \delta\left(|x'| + |x''|\right) \le \delta(2+2) = 4\delta = \varepsilon$$

$\delta = \frac{\varepsilon}{4}$ bo’lib $f(x) = x^2$ funksiyani $X = [0; 1]$ to’plamda tekis uzluksizlikka bo’ladi.




**2-misol.** $f(x) = \sin 3x$ funksiyani $R$ to’plamda tekis uzluksizlikka tekshiring.

**Yechish**: $|x' - x''| < \delta$ bo’lsin.

$$\left|f(x') - f(x'')\right| = \left|\sin 3x' - \sin 3x''\right| = \left|2\sin\frac{3(x' - x'')}{2}\cos\frac{3(x' + x'')}{2}\right| =$$

$$= 2\left|\sin\frac{3(x' - x'')}{2}\right|\left|\cos\frac{3(x' + x'')}{2}\right| \le 2\frac{3|x' - x''|}{2} < 2\frac{3\delta}{2} = 3\delta = \varepsilon.$$

Demak $\delta = \frac{\varepsilon}{3}$ bo’lib $f(x) = \sin x$ funksiya $X = R$ to’plamda tekis uzluksiz bo’ladi.

**3-misol**. $f(x) = x^2$ funksiyani $X = [0; \infty)$ to’plamda tekis uzluksizlikka tekshiring.

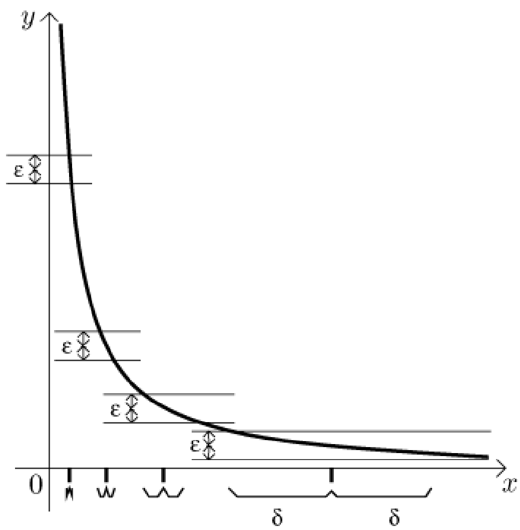
**Yechish**: $x_n' = n \in X$ va $x_n'' = n + \frac{1}{n} \in X$ nuqtalarni ko’ramiz. $|x_n' - x_n''| = \frac{1}{n} \to 0$, $|x' - x''| < \delta$ yetarlicha katta $n$ sonlar uchun o’rinli bo’ladi, ammo

$$\left|f\left(x'\right)-f\left(x''\right)\right|=\left|n^2-\left(n+\frac{1}{n}\right)^2\right|=2+\frac{1}{n^2}>2>\varepsilon$$

bo'lib $|f(x')-f(x'')|$ ayirma yetarlicha kichik emas. $f\left(x\right)=x^2$ funksiyani $X=[0;\infty)$ to'plamda tekis uzluksizlik emas.

**4-misol**. $f(x)=\frac{1}{x}$ funksiyani $X=(0;5]$ to'plamda tekis uzluksizlikka tekshiring.

**Yechish**: $x_n'=\frac{1}{n}, x_n''=\frac{1}{n+1}\in X$ bo'lsin. $|x_n'-x_n''|=\frac{1}{n(n+1)}\to 0\left(\frac{1}{n(n+1)}<\delta\right)$ bo'ladi, ammo




$\left|f\left(x'\right)-f\left(x''\right)\right|=\left|n-\left(n+1\right)\right|=1>\varepsilon$ bo'lib $|f(x')-f(x'')|$ ayirma yetarlicha kichik emas. $f\left(x\right)=\dfrac{1}{x}$ funksiyani $X=(0;5]$ to'plamda tekis uzluksizlik emas.

**Masalalar**

Tekis uzluksizlikka tekshiring.

1. $f(x)=5x+2, X=[-5;6]$;

2. $f(x)=x^3, X=[0;2]$;

3. $f(x)=\cos x, X=R$;

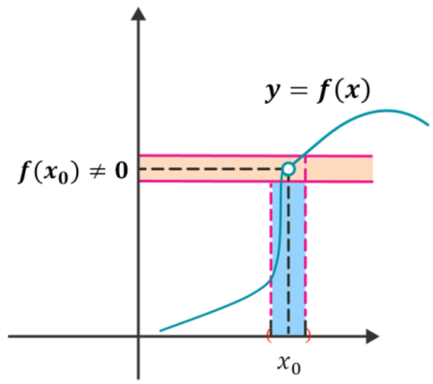
4. $f(x)=\sqrt{x}, X=\left[\frac{1}{2};\infty\right)$;

5. $f(x)=\ln x, X=(0;1]$;

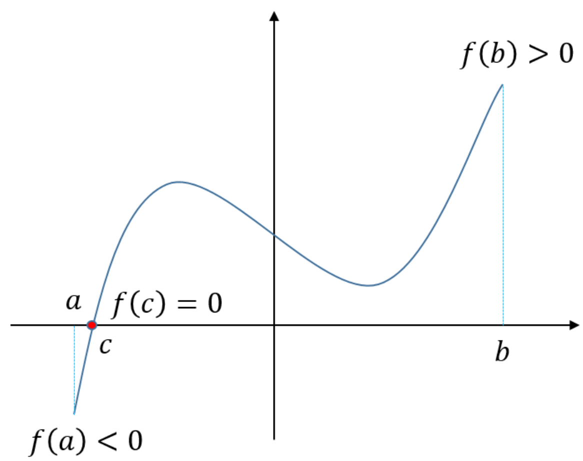
6. $f(x)=\sin x^2,\ X=[0;\infty)$.

## Uzluksiz funksiyaning xossalari

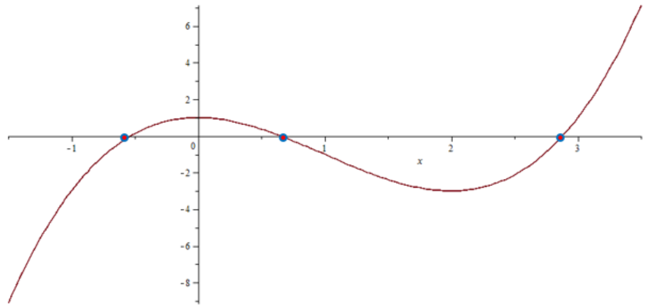
$f: E \to \mathbb{R}$ funksiya $x_0 \in E$ ichki nuqtada uzluksiz bo'lsin. Agar $f(x_0) \neq 0$ bo'lsa u holda $x_0$ nuqtaning biror atrofida ishorasini saqlaydi (yani misol uchun $f(x_0) > 0$ bo'lsa $\exists U(x_0, \delta)$ mavjudki $\forall x \in U(x_0, \delta)$ uchun $f(x) > 0$ o'rinli).




**Bolsano-Koshining 1-teoremasi**. $f \in C[a; b]$ (ya'ni $f(x)$ funksiya $[a; b]$ oraliqda uzluksiz) va $f(a)f(b) < 0$ bo'lsa u holda shunday $\exists c \in (a, b)$ son topiladiki $f(c) = 0$ o'rinli.

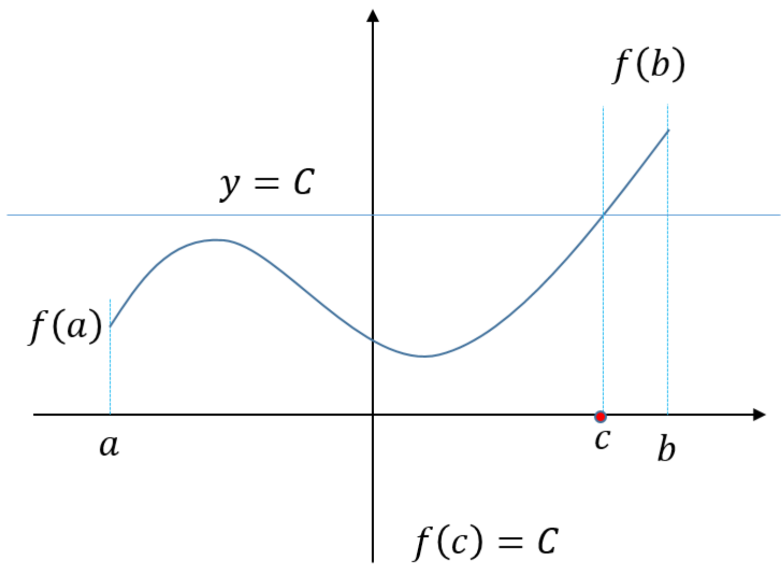



**1-misol**. $f(x) = x^3 - 3x^2 + 1$ ko'phadning ildizlari qaysi oraliqlarda yotadi?
Yechish: Ma'lumki ko'bik ko'phad hamma yerda uzluksiz. Bu funksiyaning $f(-1) = -3,\ f(0) = 1,\ f(1) = -1,\ f(2) = -3,\ f(3) = 1$ qiymatlarini ko'ramiz. $f(-1)f(0) = -3 < 0$ bo'lgani uchun Bolsano-Koshining 1-teoremasiga ko'ra shunday $x_1 \in (-1; 0)$ mavjudki $f(x_1) = 0$. Xuddi shunday quyidagi sonlarni ham topamiz:
$f(0)f(1) = -1 < 0 \Rightarrow x_2 \in (0; 1): f(x_2) = 0;$
$f(2)f(3) = -3 < 0 \Rightarrow x_3 \in (2; 3): f(x_3) = 0.$



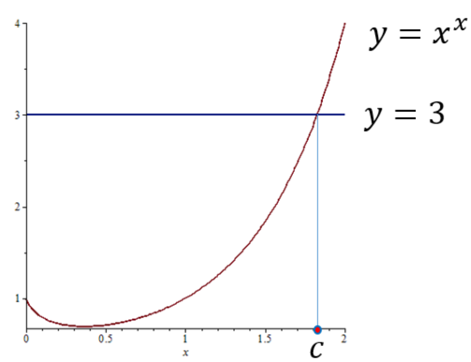

$f(x)$ – funksiya ko'bik funksiya 3 ta ildizga ega, va bu ildizlari $(-1;0)$, $(0;1)$ va $(2;3)$ oraliqlarda yotadi.

**Bolsano-Koshining 2-teoremasi**. $f \in C[a;b]$ va $f(a) < f(b)$ (yoki $f(a) > f(b)$) bo'lsa u holda $f(x)$ funksiya $f(a)$ va $f(b)$ sonlar orasidagi barcha qiymatni qabul qiladi.
Ya'ni $\forall C \in [f(a), f(b)]$ uchun $\exists c \in [a, b]: f(c) = C$

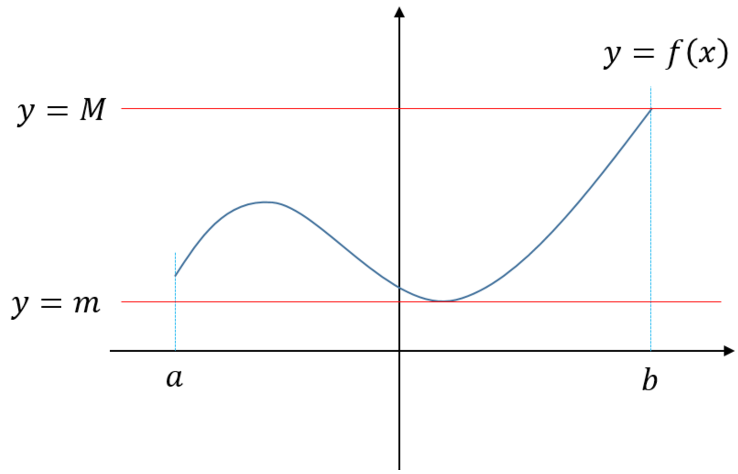



**2-misol**. $x^x = 3$ tenglamani ildizi qaysi oraliqda yotadi?

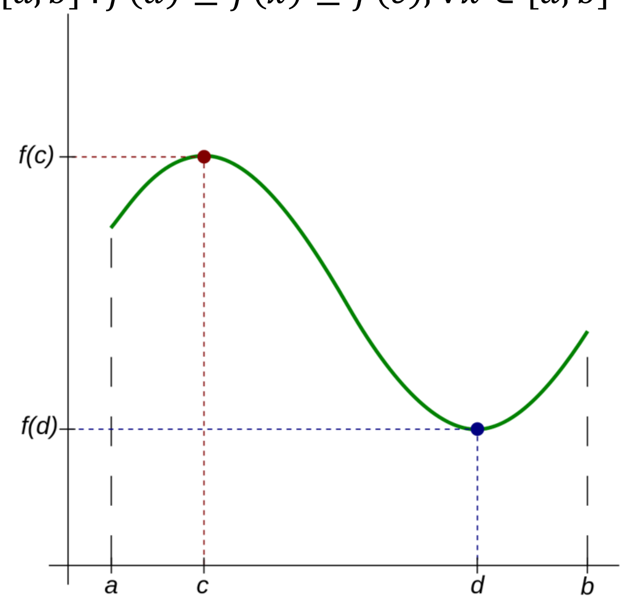
Ma'lumki $f(x) = x^x$ funksiya $[1;2]$ oraliqda uzluksiz, yani $x^x \in C[1;2]$. $f(1) = 1$, $f(2) = 4$ va $3 \in [f(1); f(2)]$ bo'lgani uchun Bolsano-Koshining 2-teoremasiga ko'ra $\exists c \in [1;2]$ mavjudki $f(c) = 3$ ya'ni $c^c = 3$. Demak $x^x = 3$ tenglama $[1;2]$ oraliqda yechimga ega.




**Veyerstassning 1-teoremasi**. $f \in C[a;b]$ bo'lsa $f(x)$ funksiya $[a, b]$ oraliqda chegaralangan.
Ya'ni $\exists m, M \in R$: $m \leq f(x) \leq M$, $\forall x \in [a;b]$.




**Veyerstassning 2-teoremasi**. $f \in C[a;b]$ bo'lsa $f(x)$ funksiya $[a,b]$ oraliqdagi yuqori aniq va quyi aniq chegaralariga erishadi.
Ya'ni shunday $\exists c, d \in [a,b] : f(d) \le f(x) \le f(c), \forall x \in [a;b]$ o'rinli.




**Masalalar**

1. $P(x) = x^{2021} - 3x + 1$ ko'phad $(0;1)$ oraliqda yechimga ega ekanligini isbotlang.
2. $x^5 - 3x = 1$ tenglama $(1;2)$ oraliqda ildizga ega ekanligini va sonlar o'qida kamida 3 ta ildizga ega ekanligini isbotlang.
3. $f \in C(\mathbb{R})$ funksiya bo'lsa $\{x \in \mathbb{R} : f(x) < c\}$ ochiq to'plam ekanligini isbotlang.
4. $f \in C(\mathbb{R})$ funksiya bo'lsa $f^{-1}(a,b) = \{x \in \mathbb{R} : f(x) \in (a,b)\}$ ochiq to'plam ekanligini isbotlang.
5. $f \in C[a,b]$ funksiya uchun shunday $c \in [a,b]$ son mavjudligini isbotlangki $f(c) = \dfrac{f(a) + f(b)}{2}$ bo'lsin.

## Funksiyaning hosilasi

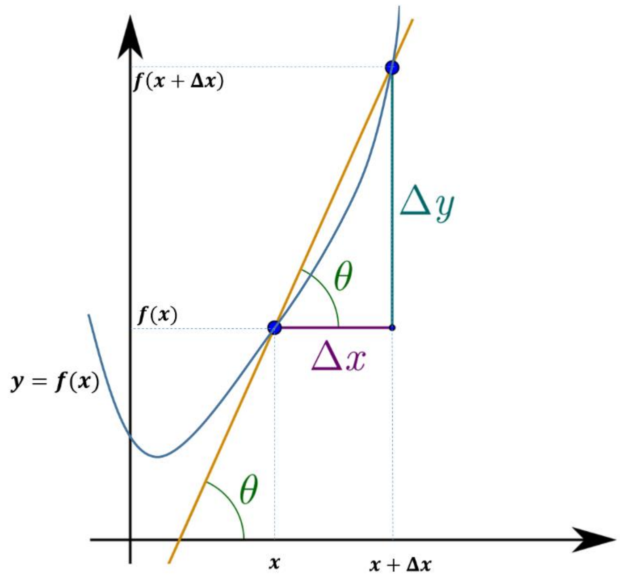
$X \subset \mathbb{R}$ to'plam berilgan bo'lsin.

**Funksiyaning hosilasi.** $f : X \to R$ *funksiyaning* $x \in X$ *ichki nuqtadagi hosilasi deb quyidagi limit qiymatiga aytiladi (agar mavjud bo'lsa):*

$$\boxed{f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}.$$

$f'(x_0)$ – geometrik ma'nosi $y = f(x)$ funksiyaga $x = x_0$ nuqtadagi urinmasini burchak koefetseyentini bildiradi:

$$k = f'(x) = tg\theta.$$




Urinma tenglamasi:

$$y - y_0 = f'(x_0)(x - x_0).$$

Ta'rif yordamida hosilani hisoshga misollar ko'ramiz.

**1-misol**: $y = \sin 2x$ .

Yechish: $\lim\limits_{\Delta x \to 0} \dfrac{y(x+\Delta x) - y(x)}{\Delta x}$ limitni topamiz.

$$(\sin 2x)' = \lim_{\Delta x \to 0} \frac{\sin 2(x+\Delta x) - \sin 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin \Delta x \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} 2\frac{\sin \Delta x}{\Delta x}\cos\left(x + \frac{\Delta x}{2}\right) = 2\cos x.$$

**2-misol**. $f(x) = x\cos x.$

$$(x\cos x)'=\lim_{\Delta x\to 0}\frac{(x+\Delta x)\cos(x+\Delta x)-x\cos x}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\big((x+\Delta x)\cos(x+\Delta x)-(x+\Delta x)\cos x\big)+\big((x+\Delta x)\cos x-x\cos x\big)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\left((x+\Delta x)\frac{\cos(x+\Delta x)-\cos x}{\Delta x}+\cos x\right)=$$

$$=\lim_{\Delta x\to 0}\left((x+\Delta x)\frac{-\sin\frac{\Delta x}{2}\sin\left(x+\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}+\cos x\right)=-x\sin x+\cos x.$$

**Masalalar**

Ta'rif yordamida funksiyaning hosilasini toping.

1. $f(x)=x^3+x;$
2. $f(x)=e^{2x};$
3. $f(x)=\ln x;$
4. $f(x)=tgx;$
5. $f(x)=xe^x;$
6. $f(x)=x^n+x^2\sin 2x$.

## **Hosila hisoblash qoidalari**:

Agar $f,g-$funksiyalar hosilaga ega bo'lsa u holda quyidagi tengliklar o'rinli:

1. $(f\pm g)'=f'\pm g';$
2. $(fg)'=f'g\pm f\,g';$
3. $\left(\frac{f}{g}\right)'=\frac{f'g-f\,g'}{g^2};$
4. $(f(g(x)))'=f'(g(x))g'(x);$
5. $y'(x)=\frac{1}{x'(y)}.$

Funksiyani hosilasini hosila hisoblash qoidalariga ko'ra topishga misollar ko'ramiz. Na'muna:

**1-misol**. $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$ .

$$f'(x)=\frac{1}{x+\sqrt{1+x^2}}\left(x+\sqrt{1+x^2}\right)'=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}\left(1+x^2\right)'\right)=$$

$$=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}.$$

**2-misol.** $f(x)=\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\frac{1-x}{1+x}.$

$$f'(x)=\left(\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\frac{1-x}{1+x}\right)'=\frac{(\arcsin x)'\sqrt{1-x^2}-\arcsin x\left(\sqrt{1-x^2}\right)'}{1-x^2}+$$

$$+\frac{1}{2}\left(\frac{1}{\frac{1-x}{1+x}}\left(\frac{1-x}{1+x}\right)'\right)=\frac{1-\arcsin x\,\frac{(1-x^2)'}{2\sqrt{1-x^2}}}{1-x^2}+\frac{1}{2}\frac{1+x}{1-x}\frac{-2}{(1+x)^2}=\frac{1+\frac{x\arcsin x}{\sqrt{1-x^2}}}{1-x^2}-$$

$$-\frac{1}{1-x^2}=\frac{\arcsin x}{\left(1-x^2\right)\sqrt{1-x^2}}=\frac{\arcsin x}{\left(1-x^2\right)^{\frac{3}{2}}}.$$

**3-misol.** $f(x)=x^{\sin x}.$

$$\ln f(x)=\ln x^{\sin x}=\sin x\ln x\Rightarrow\left(\ln f(x)\right)'=\left(\sin x\ln x\right)'\Rightarrow\frac{f'(x)}{f(x)}=\cos x\ln x+\frac{\sin x}{x}$$

$$f'(x)=f(x)\left(\cos x\ln x+\frac{\sin x}{x}\right)=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right).$$

**4-misol.** $f(x)=\left(1+3x\right)^4\left(2+5x\right)^3\sqrt[3]{5-4x}.$

$$\ln f(x)=4\ln(1+3x)+3\ln(2+5x)+\frac{1}{3}\ln(5-4x)$$

$$\frac{f'(x)}{f(x)}=\frac{12}{1+3x}+\frac{15}{2+5x}-\frac{4}{3(5-4x)}=\frac{12}{1+3x}+\frac{15}{2+5x}+\frac{4}{3(4x+5)}$$

$$f'(x)=(1+3x)^4(2+5x)^3\sqrt[3]{5-4x}\left(\frac{12}{1+3x}+\frac{15}{2+5x}+\frac{4}{3(4x+5)}\right).$$

**5-misol**. $f(x)=arctgx$.

Agar $y=arctgx$ bo'lsa $x=tgy$. Funksiyaning hosilasini topish uchun $y'(x)=\frac{1}{x'(y)}$ formuladan foydalanamiz.

$$(arctgx)'=\frac{1}{x'(y)}=\frac{1}{\frac{1}{\cos^2 y}}=\cos^2 y=\frac{1}{1+tg^2 y}=\frac{1}{1+tg^2(arctgx)}=\frac{1}{1+x^2}.$$

**Masalalar**

Funksiyaning hosilasini toping.

1. $f(x)=x\sin x$; 2. $f(x)=\frac{x}{e^x}$; 3. $f(x)=\frac{x\cos x}{\ln x}$; 4. $f(x)=\sqrt{1+x^3}$;

5. $f(x)=\sin 3x$; 6. $f(x)=\sin^5 x$; 7. $f(x)=\cos(x^3+3x)$; 8. $f(x)=\ln(\sin x)$;

9. $f(x)=\ln tg\frac{x}{2}$; 8. $f(x)=\ln\left(\arccos\frac{1}{\sqrt{x}}\right)$;

9. $f(x)=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)$; 10. $f(x)=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}$;

11. $f(x)=x^x$; 12. $f(x)=x^{e^x}$; 13. $f(x)=(1+x)^4(3x+2)^5\sqrt[4]{(3+4x)^3}$;

14. $f(x)=\frac{2}{\sqrt{a^2-b^2}}arctg\left(\sqrt{\frac{a-b}{a+b}}tg\frac{x}{2}\right),(a>b\geq 0)$.

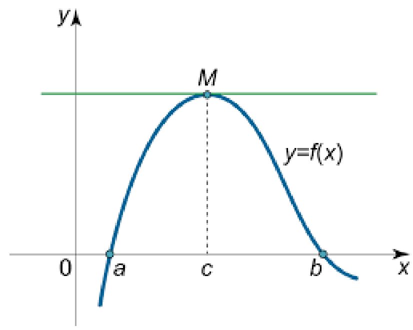
15. $y=|x|$ funksiya uchun $y'(0)-$mavjud emasligini isbotlang.

16. $y(x)=\begin{cases}x^2\sin\frac{1}{x}, x\neq 0\\ 0, x=0\end{cases}$ funksiyaning hosilasi uzulishga ega funksiya bo'lishini ko'rsating.

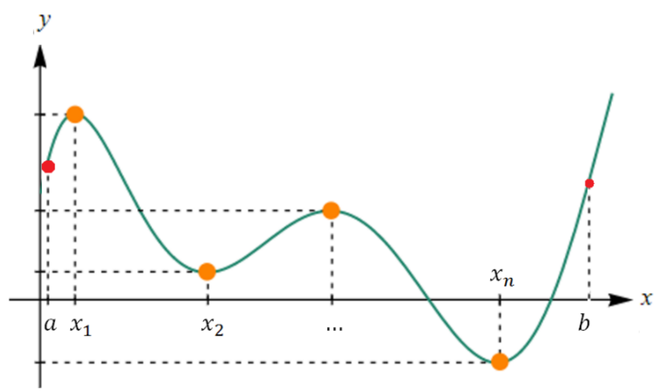
**Differensila hisobning asosiy teoremalari**

$f : C[a;b] \to \mathbb{R}$ funksiya berilgan bo'lsin.

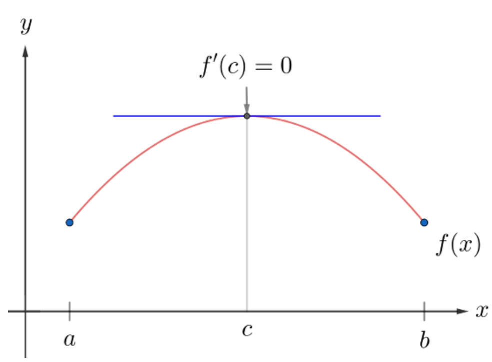
**Ferma teoramasi**. Agar $f$ funksiya $c \in (a;b)$ nuqtada maksimum (yoki minimumga) erishsa va shu nuqtada hosilaga ega bo'lsa u holda $f'(c) = 0$.




**Natija**. $f \in C^1[a;b]$ funksiyaning eng katta va eng kichik qiymatlari $M = \{f(a), f(x_1), f(x_2), ..., f(x_n), f(b)\}$ to'plamning maksimumu va minimumi bo'ladi, bu yerda $x_1, x_2, ..., x_n -$nuqtalar $f'(x) = 0$ tenglamani $(a;b)$ oraliqdagi yechimlari.



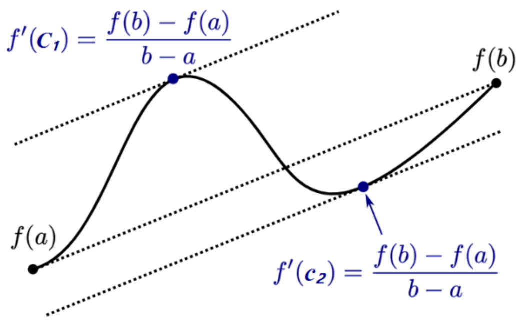

**Roll teoramasi**. Agar $f$ funksiya $[a;b]$ oraliqda uzluksiz, $(a;b)$ oraliqda hosilaga ega va $f(a) = f(b)$ bo'lsa u holda shunday $c \in (a;b)$ nuqta topiladiki $f'(c) = 0$ o'rinli.



**Lagranj teoramasi**. Agar $f$ funksiya $[a;b]$ oraliqda uzluksiz va $(a;b)$ oraliqda hosilaga ega bo'lsa u holda shunday $c\in(a;b)$ nuqta topiladiki

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

tenglik o'rinli.



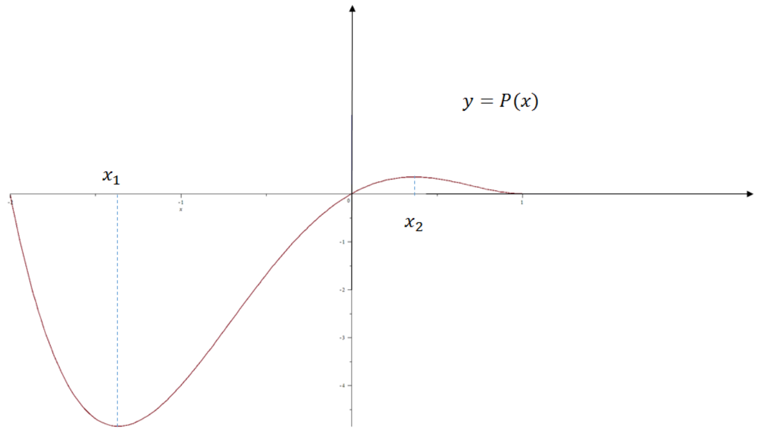

**1-misol**. $P(x)=x(x-1)^2(x+2)$ ko'phadni $[-2;1]$ oraliqdagi $|P(x)|$ ifodaning eng katta qiymatini toping.

Avval $P(x)$ ko'phadning hosilalari nolga aylanadigan qiymatlarini topamiz:

$$P'(x)=(x-1)^2(x+2)+2(x-1)x(x+2)+x(x-1)^2=4(x-1)\left(x^2+x-\frac{1}{2}\right)=0.$$

Ekstremum nuqtalari $x_1=1, x_{23}=\frac{-1\pm\sqrt{3}}{2}$ bo'lib ularni hammasi $[-2;1]$ oraliqda yotadi. Demak funksiyanign eng katta va eng kichik qiymatlari quyidagi to'plamga tegishli:

$$\left\{P(-2),P(1),P\left(\frac{-1-\sqrt{3}}{2}\right),P\left(\frac{-1+\sqrt{3}}{2}\right)\right\}$$




Demak $P_{\min} = P(x_1) = -\frac{-9+6\sqrt{3}}{2}, P_{\max} = P(x_2) = .$

**2-misol**. $0 < b < a$ haqiqiy sonlar uchun $\ln\frac{a}{b} > \frac{a-b}{a}$ tengsizlikni isbotlang.

$f(x) = \ln x$ funksiya uchun Lagranj teoremasiga ko'ra shunday $c \in [b;a]$ nuqta mavjudki quyidagi tenglik o'rinli:

$$\frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{\ln b - \ln a}{b-a} = \frac{1}{c} \Rightarrow \ln\frac{b}{a} = \frac{b-a}{c}.$$

Ya'ni $\ln\frac{a}{b} = -\ln\frac{b}{a} = \frac{a-b}{c}$. $b < c < a$ ekanligidan foydalansak $\frac{1}{c} > \frac{1}{a}$ va bu tengsizlikdan: $\ln\frac{a}{b} = \frac{a-b}{c} > \frac{a-b}{a}$ kelib chiqadi.

**3-misol**. $f \in C^1[a;b]$ (ya'ni $f' \in C[a;b]$) funksiya tekis uzluksiz ekanligini isbotlang.

$f' \in C[a;b]$ bo'lgani uchun $f'(x)$ funksiya $[a;b]$ oraliqda chegaralangan va o'zining eng katta hamda eng kichik qiymatlariga erishadi. $|f'(x)| \le M, \forall x \in [a;b]$ bo'lsin. $\forall x', x'' \in [a;b]$ sonlar uchun Lagranj teoremasiga ko'ra shunday $c \in [x';x'']$ nuqta mavjudki quyidagi tenglik o'rinli:

$$\left|\frac{f(x'')-f(x')}{x''-x'}\right| = |f'(x)| \le M \Rightarrow |f(x'')-f(x')| \le M|x'-x''|.$$

Agar $\forall\varepsilon>0$ son uchun $|x'-x''|<\delta:=\frac{\varepsilon}{M}$ bo'lganda yuqoridagi tengsizlikka ko'ra $|f(x'')-f(x')|<\varepsilon$ bo'lishi kelib chiqadi. Ya'ni $f$ funksiya $[a;b]$ oraliqda tekis uzluksiz.

**4-misol**. Agar $f$ funksiya $[a,b]$ oraliqda uzluksiz va $(a,b)$ oraliqda differensiallanuvchi hamda $a>0$ bo'lsa u holda shunday $c\in(a,b)$ son topolishini isbotlangki quyidagi tenglik bajarilsin:

$$\frac{af(b)-bf(a)}{a-b}=f(c)-cf'(c).$$

$F(x)=\frac{f(x)}{x}$ va $G(x)=\frac{1}{x}$ funksiyalar $[a,b]$ oraliqda Koshi teoremasini shartlarini bajaradi. Koshi teoremasiga ko'ra shunday $c\in(a,b)$ son topiladiki

$$\frac{\frac{f(b)}{b}-\frac{f(a)}{a}}{\frac{1}{b}-\frac{1}{a}}=\frac{\frac{cf'(c)-f(c)}{c^2}}{-\frac{1}{c^2}}$$

o'rinli. Bu tenglikni soddalashtirib $\frac{af(b)-bf(a)}{a-b}=f(c)-cf'(c)$ tenglikni hosil qilamiz.

**Masalalar**

1. $P(x)=(x-x_1)(x-x_2)...(x-x_n)$ ko'phad uchun $P'(x)=0$ tenglama $n-1$ ta haqiqiy ildizga ega ekanligini isbotlang.

2. $x,y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ sonlar uchun $|arctgx-arctgy|\le|x-y|$ tengsizlikni isbotlang.

3. $x>0$ son uchun $\frac{x}{1+x}<\ln(1+x)<x$ tengsizlikni isbotlang.

4. Agar $f\in C^1[a,b]$ funksiya uchun $f(a)=0$ bo'lsa shunday $c\in(a,b)$ son mavjudligini isbotlangki $f'(c)=f(c)$ bajarilsin. (Ko'rsatma: $F(x)=e^{-x}f(x)$ funksiya uchun Roll teoremasini qo'llang.)

5. Agar $f$ funksiya $[a,b]$ oraliqda uzluksiz va $(a,b)$ oraliqda differensiallanuvchi hamda $a>0$ bo'lsa u holda shunday $c\in(a,b)$ son topolishini isbotlangki quyidagi tenglik bajarilsin:

$$\frac{ab\big(bf(b)-af(a)\big)}{b-a}=c^2\big(f(c)+cf'(c)\big).$$

(Ko'rsatma: $F(x)=xf(x),\ G(x)=\frac{1}{x}$ funksiyalar uchun Koshi teoremasini qo'llang.)

6. Agar $f:\mathbb{R}\to\mathbb{R}$ funksiya $\forall x\in\mathbb{R}$ va $\forall h>0$ son uchun $\left|f(x+h)-f(x-h)\right|<h^2$ tengsizlik bajarilsa $f(x)\equiv const$ bo'lishini isbotlang.

## Lopital qoidalari.

**1-qoida**. Agar $g, f\in C(x_0-\delta, x_0+\delta)$ funksiyalar uchun $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=0$ bo'lib $\lim\limits_{x\to x_0}\frac{f'(x_0)}{g'(x_0)}$ – limit mavjud, $g'(x)\neq 0$ va chekli bo'lsa u holda $\lim\limits_{x\to x_0}\frac{f(x_0)}{g(x_0)}$ – limit mavjud va quyidagi tenglik o'rinli:

$$\lim_{x\to x_0}\frac{f'(x_0)}{g'(x_0)}=\lim_{x\to x_0}\frac{f(x_0)}{g(x_0)}.$$

**2-qoida**. Agar $g, f:(x_0-\delta, x_0+\delta)\to\mathbb{R}$ funksiyalar uchun $\lim\limits_{x\to x_0} f(x)=\pm\infty$, $\lim\limits_{x\to x_0} g(x)=\pm\infty$ bo'lib $\lim\limits_{x\to x_0}\frac{f'(x_0)}{g'(x_0)}$ – limit mavjud, $g'(x)\neq 0$ va chekli bo'lsa u holda $\lim\limits_{x\to x_0}\frac{f(x_0)}{g(x_0)}$ – limit mavjud va quyidagi tenglik o'rinli:

$$\lim_{x\to x_0}\frac{f'(x_0)}{g'(x_0)}=\lim_{x\to x_0}\frac{f(x_0)}{g(x_0)}.$$

**1-misol**. $\lim\limits_{x\to 0}\frac{\sin x-x}{x^3}$.

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{-\sin x}{6x}=-\frac{1}{6}.$$

**2-misol**. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x-1-x}{x\left(e^x-1\right)}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(e^x-1-x\right)'}{\left(x\left(e^x-1\right)\right)'}=\lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to 0}\frac{\left(e^x-1\right)'}{\left(e^x-1+xe^x\right)'}=\lim_{x\to 0}\frac{e^x}{2e^x+xe^x}=\frac{1}{2}.$$

**3-misol**. $\lim\limits_{x\to +0}x^x$.

$$\lim_{x\to +0}x^x=\lim_{x\to +0}e^{x\ln x}=\lim_{x\to +0}e^{\frac{\ln x}{\frac{1}{x}}}=e^{\lim\limits_{x\to +0}\frac{\ln x}{\frac{1}{x}}}=\left[\frac{\infty}{\infty}\right]=e^{\lim\limits_{x\to +0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}}=e^{\lim\limits_{x\to +0}(-x)}=e^0=1.$$

**4-misol**.

$$\lim_{x\to 0}\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln\frac{(1+x)^{\frac{1}{x}}}{e}\cdot\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{\ln\frac{(1+x)^{\frac{1}{x}}}{e}}{x}}=\left[\frac{0}{0}\right]=e^{\lim\limits_{x\to 0}\frac{\frac{1}{x}\ln(1+x)-1}{x}}=e^{\lim\limits_{x\to 0}\frac{\ln(1+x)-x}{x^2}}=\left[\frac{0}{0}\right]=$$

$$=e^{\lim\limits_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}}=\left[\frac{0}{0}\right]=e^{-\lim\limits_{x\to 0}\frac{1}{2(1+x)}}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}.$$

**Limitlarni toping**.

1. $\lim\limits_{x\to 0}\dfrac{\cos x-1-\frac{x^2}{2}}{x^4}$; 2. $\lim\limits_{x\to 0}\dfrac{tgx-x}{x^3}$; 3. $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{\ln(1+x)}\right)$; 4. $\lim\limits_{x\to +0}\sqrt{x}\ln x$;

5. $\lim\limits_{x\to +\infty}\dfrac{x^3+3x^2-3x+5}{e^x}$; 6. $\lim\limits_{x\to 0}\dfrac{\ln(\sin ax)}{\ln(\sin bx)}$; 7. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

## Yuqori tartibli hosila va differensiallar

Agar $f,g\in C^n$ funksiyalar bo’lsa u holda quyidagi tenglik o’rinli:

$$(fg)^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \frac{n(n-1)}{2!} f^{(n-2)}g'' + \frac{n(n-1)(n-2)}{3!} f^{(n-3)}g''' + \ldots + fg^{(n)},$$

bu yerda $f^{(n)}(x) := \left(f^{(n-1)}(x)\right)' -$ yuqori tartibli hosila. Ba’zi funksiyalarni yuqori tartibli hosilalarini keltiramiz:

$$(\sin x)^{(n)} = \sin\left(x + \frac{\pi n}{2}\right);$$

$$(\cos x)^{(n)} = \cos\left(x + \frac{\pi n}{2}\right);$$

$$\left(x^k\right)^{(n)} = \begin{cases} n(n-1)(n-2)\ldots(n-k+1)x^{n-k}, k < n \text{ yoki } k \notin \mathbb{N} \\ n!, k = n \\ 0, k > n \text{ va } k \in \mathbb{N} \end{cases}, \text{ bu yerda } k > 0;$$

$$\left(\frac{1}{x^k}\right)^{(n)} = (-1)^n \frac{k(k+1)\ldots(k+n-1)}{x^{n+k}}, \text{ bu yerda } k > 0;$$

$$\left(a^x\right)^{(n)} = a^x (\ln a)^n;$$

$$\left(f(kx+b)\right)^{(n)} = k^n f^{(n)}(kx+b).$$

**1-misol**. $y(x) = \left(x^2 + 5x + 3\right)\sin 3x$ bo’lsa $y^{(n)}(x)$ yuqori tartibli hosilani toping.

$f(x) = \sin 2x$ va $g(x) = x^2 + 5x + 3$ funksiyalar uchun $(fg)^{(n)}$ hosilani topish va $\left(f(kx+b)\right)^{(n)} = k^n f^{(n)}(kx+b)$ formulalarni qo’llab yuqori tartibli hosilani topamiz.

$$\left(\left(x^2 + 5x + 3\right)\sin 3x\right)^{(n)} = (\sin 3x)^{(n)}\left(x^2 + 5x + 3\right) + n(\sin 3x)^{(n-1)}(2x+5) +$$
$$+ \frac{n(n-2)}{2}(\sin 3x)^{(n-2)} 2 = 3^n\left(x^2 + 5x + 3\right)\sin\left(3x + \frac{\pi n}{2}\right) +$$
$$3^{n-1}n(2x+5)\sin\left(3x + \frac{\pi(n-1)}{2}\right) + 3^{n-2}n(n-2)\sin\left(3x + \frac{\pi(n-2)}{2}\right).$$

**2-misol**. $y = \frac{1}{\sqrt{2-3x}}$ bo’lsa $y^{(n)}(x)$ yuqori tartibli hosilani toping.

Agar $\left((kx+b)^a\right)^{(n)} = a(a-1)(a-2)\dots(a-n+1)k^n(kx+b)^{a-n}$ formuladan foydalansak u holda hosilasi quyidagicha:

$$y^{(n)} = \left((2-3x)^{-\frac{1}{2}}\right)^{(n)} = -\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{2n-1}{2}\right)(-3)^n(2-3x)^{-\frac{1}{2}-n} =$$

$$= \frac{3^n(2n-1)!!}{2^n}\frac{1}{(2-3x)^n\sqrt{2-3x}}.$$

**3-misol**. $y(x) = \dfrac{x^2}{\sqrt{1-2x}}$ bo'lsa $y^{(n)}(x)$ yuqori tartibli hosilani toping.

Avval funksiyani quyidagicha ko'rinishda yozib olamiz:

$$y(x) = \frac{x^2}{\sqrt{1-2x}} = \frac{4x^2}{4\sqrt{1-2x}} = \frac{4x^2-4x+1}{4\sqrt{1-2x}} + \frac{4x-2}{4\sqrt{1-2x}} + \frac{1}{4\sqrt{1-2x}} =$$

$$= \frac{1}{4}(1-2x)^{\frac{3}{2}} - \frac{1}{2}(1-2x)^{\frac{1}{2}} + \frac{1}{2}(1-2x)^{-\frac{1}{2}}.$$

$y^{(n)}(x) = \dfrac{1}{4}\left((1-2x)^{\frac{3}{2}}\right)^{(n)} - \dfrac{1}{2}\left((1-2x)^{\frac{1}{2}}\right)^{(n)} + \dfrac{1}{2}\left((1-2x)^{-\frac{1}{2}}\right)^{(n)}$ tenglikdan foydalanib har bir hosilani 2-misoldagi kabi soddalashtirib quyidagi javobni olamiz:

$$y^{(n)}(x) = \frac{(2n-5)!!\left(3x^2-2nx+n^2-n\right)}{(1-2x)^{n+\frac{1}{2}}}.$$

**4-misol**. Agar $y(x) = e^{ax}\cos bx$ bo'lsa $y^{(n)}(x)$ yuqori tartibli hosilani toping.

Funksiyani hosilalarini topamiz:

$$y'(x) = ae^{ax}\cos bx - be^{ax}\sin bx = e^{ax}(a\cos bx - b\sin bx) = e^{ax}\sqrt{a^2+b^2}\cos(bx+\varphi),$$

bu yerda $\cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}, \sin\varphi = \dfrac{b}{\sqrt{a^2+b^2}}$.

$$y''(x) = e^{ax}\sqrt{a^2+b^2}\left(a\cos(bx+\varphi) - b\sin(bx+\varphi)\right) = e^{ax}\sqrt{a^2+b^2}^{\,2}\cos(bx+2\varphi);$$

$$y'''(x) = e^{ax}\sqrt{a^2+b^2}^{\,2}\left(a\cos(bx+2\varphi) - b\sin(bx+2\varphi)\right) = e^{ax}\sqrt{a^2+b^2}^{\,3}\cos(bx+3\varphi).$$

Matematika induksiya usuli orqali $y^{(n)}(x)=e^{ax}\sqrt{a^2+b^2}^{\,n}\cos(bx+n\varphi)$ tenglik o'rinli bo'lishini ko'rsatish mumkin.

**Masalalar**

Quyidagi funksiyalarni $n-$tartibli hosilalarini toping.

1. $y=a^{kx+b}$; 2. $y=\ln 2x$; 3. $y=\frac{1}{x-1}$; 4. $y=xe^x$; 5. $y=\frac{5x+4}{x^2-5x+6}$ (Ko'rsatma $\frac{5x+4}{x^2-5x+6}=\frac{A}{2-x}+\frac{B}{3-x}$ ko'rinishida yozib oling); 6. $y=x^2\cos 3x$; 7. $y=\frac{x}{\sqrt{1-x}}$;

8. $y=\sin 3x\cos 5x$; 9. $y=x\ln\frac{1+x}{1-x}$; 10. $y=e^{ax}\sin bx$; 11. $y=\sin^4 x+\cos^4 x$.

12. $\left(\frac{\ln x}{x}\right)^{(n)}=\frac{(-1)^n n!}{x^{n+1}}\left(\ln x-1-\frac{1}{2}-\frac{1}{3}-...-\frac{1}{n}\right)$ tenglikni isbotlang.

## Teylor formulasi

Agar $f\in C^{(n+1)}(x_0-\delta,x_0+\delta)$ bo'lsa u holda quyidagi tenglik o'rinli:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\ldots+$$

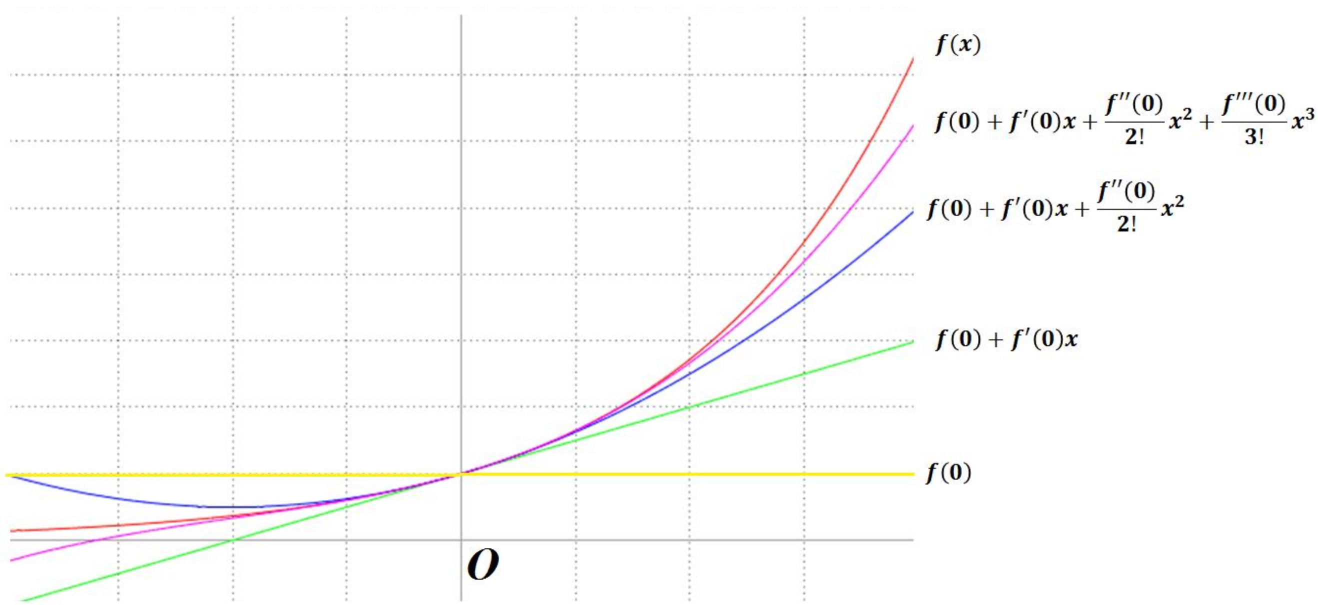
$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o\left((x-x_0)^n\right),$$

bu yerda $o\left((x-x_0)^n\right)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^n$ va $c-x$ va $x_0$ orasidagi son.

Bu formula $f(x)$ funksiyaning $x=x_0$ nuqtadagi Teylor formulasi deyiladi. Yig'indi ko'rinishida quyidagicha yozish mumkin:

$$f(x)=\sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+o\left((x-x_0)^n\right).$$

Agar $x_0=0$ bo'lsa formula $f(x)=\sum_{k=1}^{n}\frac{f^{(k)}(0)}{k!}x^k+o(x^n)$ ko'rinishida bo'ladi.




Bu hol uchun quyida ba'zi elementar funksiyalar uchun yoyilmani keltiramiz:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+o\left(x^n\right);$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+o\left(x^{2n+2}\right);$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots+\frac{(-1)^n x^{2n}}{(2n)!}+o\left(x^{2n+1}\right);$$

$$(1+x)^a=1+\frac{a}{1!}x+\frac{a(a-1)}{2!}x^2+\ldots+\frac{a(a-1)(a-2)\ldots(a-n+1)}{n!}x^n+o\left(x^n\right);$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+o\left(x^n\right);$$

$$-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+o\left(x^n\right).$$

**1-misol**. $f(x)=\sqrt{x}$ funksiyani $x_0=1$ nuqtada Teylor formulasini yozing.

$f(1), f'(1), f''(1),\ldots, f^{(n)}(1)$ qiymatlarni topamiz:

$$f(x)=\sqrt{x}\Rightarrow f(1)=1;$$

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}} \Rightarrow f'(1)=\frac{1}{2};$$

$$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} \Rightarrow f''(1)=-\frac{1}{4};$$

$$f'''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} \Rightarrow f'''(1)=\frac{3}{8};$$

$$f^{IV}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} \Rightarrow f^{IV}(1)=-\frac{15}{16};$$

Umumiy holda $f^{(n)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{2n-3}{2}\right)x^{-\frac{2n-1}{2}}$ qonuniyatni sezish va uni matematik induksiya metodi yordamida oson ko’rsatish mumkin. Bu holda $f^{(n)}(1)=\frac{(-1)^{n-1}(2n-3)!!}{2^n}$ va Teylor formulasiga ko’ra ushbu

$$\sqrt{x}=1+\frac{1}{2\cdot 1!}(x-1)-\frac{1}{4\cdot 2!}(x-1)^2+\frac{3}{8\cdot 3!}(x-1)^3+\ldots+\frac{(-1)^{n-1}(2n-3)!!}{2^n n!}(x-1)^n+o\left((x-1)^n\right)$$

tenglik o’rinli.

**2-misol**. $\sin 1$ sonni qiymatini verguldan keyin 5 ta raqam aniqligida hisoblang.

Malumki:

$$\left|\sin x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right|=o\left(x^{2n+2}\right)=\frac{\left|(\sin x)^{(2n+2)}\Big|_{x=c}\right|}{(2n+2)!}\left|x^n\right|\le\frac{\left|x^n\right|}{(2n+2)!}.$$

U holda quyidai tengsizlik bajarilishi kerak:

$$\left|\sin 1-1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots+\frac{(-1)^n}{(2n+1)!}\right|\le\frac{1}{(2n+2)!}<\frac{1}{10000}$$

ya'ni $(2n+2)!>10000$. $7!<10000<8!$ bo’lgani uchun $n\ge 8$ bo’lib berilgan aniqlikka erishish uchun $x^8=1^8$ gacha bo’lgan hadlarini olish yetarli. Biroq $x^8$ oldidagi koefetseyent 0 ga teng. Shuning uchun $\sin 1\approx 1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}\approx 0.84147.$

**4-misol**. Limitni toping: $\lim\limits_{x\to 0}\dfrac{e^x - x - \cos x}{x^2}$.

Ma’lumki $e^x = 1 + x + \dfrac{x^2}{2} + o\left(x^2\right)$ va $\cos x = 1 - \dfrac{x^2}{2} + o\left(x^3\right) = 1 - \dfrac{x^2}{2} + o\left(x^2\right)$. Bu tengliklardan foydalanamiz:

$$\lim_{x\to 0}\frac{e^x - x - \cos x}{x^2} = \lim_{x\to 0}\frac{1 + x + \dfrac{x^2}{2} + o\left(x^2\right) - x - 1 + \dfrac{x^2}{2} + o\left(x^2\right)}{x^2} = \lim_{x\to 0}\left(1 + o(1)\right) = 1.$$

**5-misol**. Limitni toping: $\lim\limits_{x\to 0}\dfrac{\sin\left(\sin x\left(\sin x\left(\sin x\right)\right)\right) - x}{x^3}$.

Ma’lumki $\sin x = x - \dfrac{x^3}{6} + o\left(x^3\right)$. Shuning uchun quyidagi tengliklar o’rinli:

$$\sin\left(\sin x\right) = \sin\left( = x - \frac{x^3}{6} + o\left(x^3\right)\right) = x - \frac{x^3}{6} + o\left(x^3\right) - \frac{\left(x - o(x)\right)^3}{6} = x - 2\cdot\frac{x^3}{6} + o\left(x^3\right);$$

$$\sin\left(\sin x\left(\sin x\right)\right) = \sin\left(x - 2\cdot\frac{x^3}{6} + o\left(x^3\right)\right) = x - 2\cdot\frac{x^3}{6} + o\left(x^3\right) - \frac{\left(x - o(x)\right)^3}{6} =$$

$$= x - 3\cdot\frac{x^3}{6} + o\left(x^3\right);$$

$$\sin\left(\sin x\left(\sin x\left(\sin x\right)\right)\right) = \sin\left(x - 3\cdot\frac{x^3}{6} + o\left(x^3\right)\right) = x - 3\cdot\frac{x^3}{6} + o\left(x^3\right) - \frac{\left(x - o(x)\right)^3}{6} =$$

$$= x - 4\cdot\frac{x^3}{6} + o\left(x^3\right) = x - \frac{2x^3}{3} + o\left(x^3\right).$$

Endi limitni hisoblaymiz:

$$\lim_{x\to 0}\frac{\sin\left(\sin x\left(\sin x\left(\sin x\right)\right)\right) - x}{x^3} = \lim_{x\to 0}\frac{x - \dfrac{2x^3}{3} + o\left(x^3\right) - x}{x^3} = \lim_{x\to 0}\left(-\frac{2}{3} + o(1)\right) = -\frac{2}{3}.$$

**6-misol**. $f\left(x\right) = \sqrt[3]{\sin x^3}$ funksiyani $x = 0$ nuqta atrofida $x^{13}$ hadigacha Teylor formulasini yozing va $f^{(13)}\left(0\right)$ toping.

$x^3 = t$ belgilash kiritamiz. $o\left(x^{13}\right)$ hadigacha yoyish uchun $\sin x^3 = \sin t = t - \frac{t^3}{6} + \frac{t^5}{120} + o\left(t^6\right) = t\left(1 - \frac{t^2}{5} + \frac{t^4}{120} + o\left(t^5\right)\right)$ olish yetarli. U holda

$$f(x) = \sqrt[3]{\sin t} = t^{\frac{1}{3}}\left(1 - \frac{t^2}{5} + \frac{t^4}{120} + o\left(t^5\right)\right)^{\frac{1}{3}} = \left[\alpha(t) = -\frac{t^2}{5} + \frac{t^4}{120} + o\left(t^5\right)\right] = t^{\frac{1}{3}}\left(1 + \alpha(t)\right)^{\frac{1}{3}} =$$

$$= x\left(1 + \frac{\alpha(t)}{3} - \frac{\alpha(t)^2}{9} + o\left(\alpha(t)^2\right)\right) = x\left(1 + \frac{-\frac{t^2}{6} + \frac{t^4}{120}}{3} - \frac{\left(-\frac{t^2}{6} + \frac{t^4}{120}\right)^2}{9} + o\left(t^5\right)\right) =$$

$$= x\left(1 - \frac{t^2}{18} - \frac{t^4}{3240} + o\left(t^5\right)\right) = x\left(1 - \frac{x^6}{18} - \frac{x^{12}}{3240} + o\left(x^{15}\right)\right) = x - \frac{x^7}{18} - \frac{x^{13}}{3240} + o\left(x^{16}\right).$$

Bu tenglikdan $c_{13} = \frac{f^{(13)}(0)}{13!} = -\frac{1}{3240}$ kelib chiqadi. Demak $f^{(13)}(0) = -\frac{13!}{3240} = -1921920.$

Javob: $\sqrt[3]{\sin x^3} = x - \frac{x^7}{18} - \frac{x^{13}}{3240} + o\left(x^{16}\right)$ va $f^{(13)}(0) = -1921920.$

**7-misol**. $f \in C^2[0;1]$ funksiya uchun $f(0) = f(1) = 0$ va $\forall x \in [0;1]$ uchun $\left|f''(x)\right| \le M$ tengsizlik bajarilsa u holda $\forall x \in [0;1]$ uchun $\left|f'(x)\right| \le \frac{M}{2}$ tengsizlikni isbotlang.

$t$ nuqta atrofida funksiyani Teylor formulasini qo'laymiz:

$$f(x) = f(t) + f'(t)(x - t) + \frac{f''(c(x))}{2!}(x - t)^2.$$

Agar $x=0$ bo'lsa $0=f(t)-tf'(t)+\frac{f''(c(0))}{2!}t^2$. $x=1$ bo'lsa u holda $0=f(t)+f'(t)(1-t)+\frac{f''(c(1))}{2!}(t-1)^2$ bo'lib oxirgi ikki tenglikni qo'shib $f'(t)=\frac{1}{2}\left(f''(c(0))t^2-f''(c(1))\frac{(1-t^2)}{2}\right)$ tenglikka ega bo'lamiz. Bu tenglikdan

$$\left|f'(t)\right|=\frac{1}{2}\left|f''(c(0))t^2-f''(c(1))\frac{(1-t^2)}{2}\right|\le$$

$$\le\frac{1}{2}\left(\left|f''(c(0))\right|t^2+\left|f''(c(1))\right|\frac{(1-t^2)}{2}\right)\le\frac{1}{2}\left(Mt^2+M\frac{(1-t^2)}{2}\right)=\frac{M}{2}\left(2t^2-2t+1\right)$$

tengsizlikni hosil qilamiz. $t\in[0;1]$ oraliqda $0\le 2t^2-2t+1\le 1$ bo'lgani uchun $\left|f'(t)\right|\le\frac{M}{2}$ tengsizlik o'rinli ekanligi kelib chiqadi.

**Masalalar**

1. $f(x)=e^{2x}$ funksiyani $x=\frac{1}{2}$ nuqta atrofida $n=6$ hadigacha Teylor formulasini yozing.

2. $f(x)=\sqrt[3]{x}$ funksiyani $x=\frac{1}{2}$ nuqta atrofida $n=5$ hadigacha Teylor formulasini yozing.

3. $f(x)=\frac{x}{(1-x)^2}$ funksiyani $x=0$ nuqta atrofida $n=7$ hadigacha Teylor formulasini yozing.

4. $f(x)=tgx$ funksiyani $x=0$ nuqta atrofida $n=3$ hadigacha Teylor formulasini yozing.

5. $f(x)=\sin 2x$ funksiyani $x=\frac{\pi}{2}$ nuqta atrofida $n=6$ hadigacha Teylor formulasini yozing.

6. $f(x)=x^x$ funksiyani $x=1$ nuqta atrofida $n=3$ hadigacha Teylor formulasini yozing.

7. Limitlarni hisoblang. a. $\lim\limits_{x\to 0}\frac{\ln(1+x)-\sin x}{x^2}$; b. $\lim\limits_{x\to 0}\frac{\sin x - x\cos x}{x^3}$;

c. $\lim\limits_{x\to 0}\frac{\sin(tgx)-tg(\sin x)}{x^3}$ (Ko'rsatma: $tgx = x+\frac{x^3}{3}+o(x^3)$ dan foydalaning).

8. $f(x)=\ln\left(\frac{1-x}{1+x}\right)$ funksiyani $x=0$ nuqta atrofida $x^{2n+1}$ hadigacha Teylor formulasini yozing.

9. $f(x)=\frac{\sin x}{x}$ funksiyani $x=0$ nuqta atrofida $x^{2n}$ hadigacha Teylor formulasini yozing.

10. $f(x)=(1+x)^{\frac{1}{x}}$ funksiyani $x=0$ nuqta atrofida $x^2$ hadigacha Teylor formulasini yozing.

11. $f(x)=\frac{x}{e^x-1}$ funksiyani $x=0$ nuqta atrofida $x^4$ hadigacha Teylor formulasini yozing.

12. $f(x)=\ln\frac{\sin x}{x}$ funksiyani $x=0$ nuqta atrofida $x^6$ hadigacha Teylor formulasini yozing.

13. $\sin 18^0$ qiymatini verguldan keyin 5 ta raqam aniqligida hisoblang.

14. Agar $f\in C^2(\mathbb{R})$ funksiya uchun $M_0=\sup\limits_{x\in R}|f(x)|, M_1=\sup\limits_{x\in R}|f'(x)|$ va $M_2=\sup\limits_{x\in R}|f''(x)|$ bo'lsa u holda $M_1^2\le 2M_0M_2$ tengsizlikni isbotlang. (Ko'rsatma $0\le|f(0)|\le|f(x)|+|f'(x)||x|+|f''(x)|\frac{|x|^2}{2}\le M_0+M_1y+M_2\frac{y^2}{2}$ tengsizlikni hosil qilib $M_0+M_1y+M_2\frac{y^2}{2}$ kavdrat uchhad musbatligidan foydalaning, bu yerda $y=|x|$.)